Momentum Transfer in Fluids
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I welcome you to this lecture on Momentum Transfer and Fluid. Discussing dimensional analysis and similitude. In particular, we talked about how to get dimensionless number and we proposed a scheme which goes by the name Buckingham π theorem. So, that using that theorem we try to find out if we have several parameters for example, pressure drop and diameter, velocity, density, viscosity. So, how do we select these parameters and find out which one within themselves will form a dimensionless number and how to do it in a scientific way, how to do it in a systematic way.

So, that is something which we were discussing at the end of last class which goes by the name Buckingham π theorem. I will continue this lecture now. So, what we discussed at the end of last class is that we are talking about drag, problem of a drag where drag force would be depending on diameter, velocity, density, viscosity. So, we have five such parameters, we have listed them and out of that we left F and μ and remaining ρ, v and D we selected and we wanted that to be a recurring set.

So, now, we have to find out what is the dimensionless number corresponding to this. So, what we write here is I said there are two dimensionless numbers involved one is Π_1 another is Π_2 and I have ρ, v and D these are the recurring variables we have selected following certain property following certain characteristics and I have discussed that at the end of last class. So, following those characteristics we have come up with this $\Pi_1 = \rho^a V^b D^c F$ and this is the here I see the F term and similarly ρ v D I have the exponent EFG with the μ term. So, what we see here is if I write Π_1 now Π_1 is a dimensionless number. Buckingham π theorem states that number of variables minus the number of fundamental dimensions.

So, 5 minus 3 so, there would be two dimensionless numbers. So, these are here. So, you have ρ to the power A. So, Π_1 is dimensionless number. So, ρ is ρ to the power A.

So, ρ appears here as mass per length cube density is kg per meter cube to the power A length per time to the power B velocity to the power B diameter D which has a unit of length to the power C, and then we have the unit of F which is mass unit if the force is mass into acceleration M into A and A is length per time square. So, that is why we have $\frac{M\,L}{t^2}$ and that is supposed to be a dimensionless number. So, this is equal to $M^0 L^0 t^0$.

$$\left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \left(\frac{M\,L}{t^2}\right) = M^0 L^0 t^0$$

So, we if we now equate if we now try to find out we see here M to the power a here and again I have here another M to the power 1. So, M to the power a into M to the power 1 and that and on this side I see M to the power 0.

So, we have to have a M to the power a into M to the power 1 is a plus 1. So, M to the power a plus 1. So, M to the power a plus 1 has to be equal to M to the power 0 or a plus 1 has to be equal to 0 or a has to be equal to minus 1. Similarly, b when it comes to length let us say length has length is everywhere length is here L to the power minus 3 a. So, L to the power minus 3 a from this first term L to the power b from the second term L to the power c from the third term and m L by t square.

So, L to the power 1 from the fourth term that is equal to L to the power 0. So, that means, -3a+b+c = 0. So, that is on one relation. When it comes to t I see here t to the power B here t to the power minus B here. So, T to the power minus B here and T to the power minus 2 here t is in denominator in both cases that is equal to t to the power 0.

So, that means, I have –b-2=0. So, this tells me that b=-2. So, I have a is equal to minus 1, b is equal to minus 2 put these values here and you will get what is the value of c from this expression. So, you got what is a, b, c you repeat the same exercise now with Π_2 repeat the same exercise with Π_2, but instead of F now you have μ. It would be again ρ to the power e.

So, that means, it would be M by L cube to the power e, L by t to the power F and L to the power G that is diameter to the power G and then μ. μ is kg per meter second or gram per centimeter second or it would be M divided by LT that is equal to M to the power 0 because this is dimensionless L to the power 0, t to the power 0. So, now, again you will have a set of relations with e, f and g again you will have what again you will have m to the power E here and m to the power 1. So, here you will get here you will get e plus 1 that is equal to 0 which implies e is equal to minus 1 then again with L you will have a long relation which is -3e+f+g that is equal to 0 from equating with L and with t you will have –f-1=0. So, from here you get e is equal to minus 1 from here you get f is equal to minus 1 and then putting those values of e and f you will get the other value of g.

So, this will so, moment you know these exponents a, b, c and e f g. So, you can find out what if for example, when you know the a b and c then you immediately see this is the dimensionless number because you put a b and c and then you get you get what a b and c you have. So, what is Π_1? I said Π_1 is ρ to the power a, v to the power b, D to the power c into F. So, put this a b and c. a is minus 1, b is minus 2. So, you have so, this Π_1 would be equal to F is there a is minus 1.

So, it is ρ is in the denominator ρ to the power minus 1. So, it is going to the denominator b is minus 2 and b is with velocity. So, V is V to the power 2 it is going into the denominator and similarly you will find c is also equal to minus 2. So, you will have it is going to the denominator.

So, you will have ρ to the power minus 1, v to the power minus 2 and then D to the power minus 2 into F. So, that is a dimensionless number or you put go let them go to the denominator $\frac{F}{\rho V^2 D^2}$. So, that is a dimensionless number. So, this is how you get the dimensionless number here. Similarly, with μ you will get the Π_2 with Π_2 you will get another dimensionless number.

$$\Pi_1 = \frac{F}{\rho V^2 D^2}$$

So, now, you know that there are these two dimensionless numbers and now you relate one dimensionless number against the other one dimensionless number in the y axis which one will you put to y axis the one that has F in it. So, that means, this one because you want F is the y axis F is something which you want to know as a function of other terms. So, dimensionless number involving F goes to the y axis and the other dimensionless number goes to the x axis. Now, you perform experiments the experiments that are feasible, which means you work with the same sphere, same fluid, same density, viscosity, etcetera, only change the velocity, which is in your control. It can do it very easily using some pump settings. Etcetera. So, this simplifies the whole thing this helps us in handling it in a very clean and neat way.

So, this is what is essentially Buckingham π theorem. You have to be very careful while selecting the recurring set. There are certain rules that we pointed out, what to consider as part of the recurring set and what not to. So, please follow them very carefully. Now, there are several Reynolds numbers which are important in many different contexts. For example, Reynolds number $\frac{\rho VD}{\mu}$, Reynolds number gives you the inertia force by viscous force.

What does it mean? Reynolds number gives you inertia force by viscous force that means, I have a fluid flowing through a pipe and I have introduced a color in that flow some dye I have continuously injecting at a very small location and then I see that the color travels like a straight line through this tube water is flowing and the color travels like a straight line. So, I know that here, in this case, one layer is sliding against the other; there is no cross current involved, no eddies involved, either one layer sliding against the other, or it could be plug flow, but there is no cross eddies forming. But at some point, when you continue to increase the velocity, you find that this color becomes wavy, and at one point, you find that this color becomes it when it starts forming. It is all getting mixed up. So, we say that we have had a transition from a laminar flow to turbulent flow. That means, the flow that was laminar means flow when one layer was sliding against the other.

So, you expect that there is no mixing between the layers, but then at one point we see that that is not no longer maintained. So, what did you do differently? You have increased the velocity. So, you have increased the velocity means $\frac{\rho V D}{\mu}$. So, you have increased the velocity. You have increased the velocity means you have increased the inertia with respect to the shear force because this denominator this is inertial force by viscous force.

So, inertia force when that increases you will end up with having this type of chaotic flow. You need not have to do only with the velocity you do not have to increase the velocity you can retain the same velocity, but increase the diameter. So, I had a tube of some dimension of some velocity and the same velocity I maintained by changing the flow rate, but I increase the diameter of the pipe. I will find that the flow becomes unstable here in this case as well because when one layer slides against the other what is what is how they are sliding against the other the at the wall velocity is 0 little bit away from the wall velocity little bit away from the wall further the velocity and then by the time and then soon the effect of other wall will also come into play. So, you will have a velocity boundary layer developing and here also boundary layer developing they will merge and you will find a laminar flow over the entire cross section.

But now you have in this case if the diameter is too large you will find that this build-up of boundary layer at one point these boundary layer becomes too large for one layer sliding against the other and the effect of wall dominating and forcing one layer sliding against the other in the viscous mode that no longer is valid recirculation sets in. So, when the diameter is too large you will end into those issues. So, the flow becomes unstable. It is not one layer sliding against the other over the entire cross-section; instead, you have some kind of cross-flow happening, to recycle some river flow in reverse directions. So, that is onset of turbulence. So, you can simply by changing diameter also you can change the Reynolds number.

So, this Reynolds number signifies the interplay of inertia and viscous force, and in turn, this signifies the inertia. It signifies whether the flow would be laminar flow or flow would be chaotic. Similarly, there are several other dimensionless numbers that are there and, say, the Euler number we mentioned. Euler number gives you $Eu = \frac{\Delta p}{\frac{1}{2}\rho V^2}$. So, or you can call it the velocity head and Δ p is the pressure head that you have. So, you when you lose kinetic head kinetic energy and in turn you get pressure head you will probably talk about Euler number and this happens all the time.

Suppose I have a converging channel what we see is that as the cross section area decreases the velocity increases. So, that means the kinetic head kinetic energy increases the kinetic energy content increases at the expense of what the pressure head. So, pressure head decreases. So, this Euler number gives you an interplay between the pressure and the kinetic energy. Similarly, you have cavitation number say I can think of there are other say for example, I tell you something called capillary number.

Capillary number is given by µ u by σ. I can think of let us say I have a two phase flow I have another phase coming in here. I have a droplet let us say I have a droplet and this droplet will have this droplet will retain its shape. Let us say I have this droplet and I have a fluid flow taking place inside a channel. So, here one layer is sliding against the other.

So, it will tend to shear the droplet. So, that one layer sliding against the other means I have a velocity profile like this. So, this is the velocity profile. So, that means any droplet the droplet will also be instead of having this shape the droplet will also take a shape like this. This would be the shape of the droplet instead of sphere. So, droplet will be also be stretched because of this field because of the stress field.

Step 4

Select a set of r dimensional parameters that includes all the primary dimensions

Example: For drag on a sphere (m = r = 3)

Select ρ, V, D

Step 5

Set up dimensional equations, combining the parameters selected in step 4 with each of the other parameters in turn, to form dimensionless groups

There will be (n-m) equations
Example: For drag on a sphere

$$\Pi_1 = \rho^a V^b D^c F$$

$$\Pi_2 = \rho^e V^f D^g \mu$$

For drag on a sphere

$$\left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b (L)^c \left(\frac{M\,L}{t^2}\right) = M^0 L^0 t^0$$

$$\Pi_1 = \frac{F}{\rho V^2 D^2}$$

Step 6

Check to see that each group obtained is dimensionless

Example: For drag on a sphere

$$F\frac{L^4}{Ft^2}\left(\frac{t}{L}\right)^2\frac{1}{L^2} = 1$$

Significant Dimensionless Groups in Fluid Mechanics

- ✓ Reynolds number, $Re = \frac{\rho V D}{\mu} = \frac{VL}{^{\mu}/_{m}}$
- ✓ Mach number, $M = \frac{V}{c}$
- ✓ Froude number, $Fr = \frac{V}{\sqrt{gL}}$
- ✓ Weber number, $W_e = \frac{\rho V^2 L}{\sigma}$
- ✓ Euler number, $Eu = \frac{\Delta p}{\frac{1}{2}\rho V^2}$
- ✓ Cavitation Number, $Ca = \frac{p - p_0}{\frac{1}{2}\rho V^2}$

So, now droplet, on the other hand, if you leave it, it will tend to form a sphere again because of surface tension force. In this context you may see the importance of something called Laplace pressure where you have a sphere let us say a sphere would be if this is the sphere this if you cut a cut the sphere you will find that there is a surface tension force acting and this the pressure inside this droplet and pressure outside they are different. And this surface tension force defines how much is the pressure outside and how much is the pressure inside and they have the inside the pressure is more outside the pressure is less. So, if you take this sphere let us say and squeeze it and then leave it you will find it will go back to its spherical shape because that is how it can it would be it has to it has to satisfy the Laplace equation. So, wherever you have squeezed, wherever you change the radius of curvature, you will find the reverse flow takes place such that the radius of curvature becomes uniform again.

So, you have the surface tension force that ensures that the sphere remains it retains its spherical shape, whereas, when you expose this to this type of shear environment, the droplet will be stretched in the form of a parabolic strip of some kind. So, under such a situation, the droplet tends to disintegrate. The droplet is supposed to disintegrate. On the other hand, the droplet is held together, and the droplet disintegrates the resisting forces because of surface tension. So, now, it would be on one hand you are imposing a shear stress, which will tend to disintegrate the droplet into finer droplets when you suspend a droplet in a moving flow inside a channel, a viscous flow inside a channel. On the other hand, the surface tension force tends to tend to resist that. If you want to see the interplay between the two, then you go by the capillary number, which goes by the μ, which goes by these variables μ u by σ.

μ is the viscosity u is the velocity of the fluid that we are talking about and σ is the surface tension. So, it is the viscous force by surface tension force. If surface tension force if σ is large then you have capillary number which is small. So, if the capillary number is small, surface tension will dominate, whereas, when the capillary number is large μ will dominate, the viscous force will dominate. So, depending on what capillary if capillary number is large that means, viscous force is going to dominate and that means, the particle will disintegrate because σ is insignificant compared to that.

Whereas, if σ becomes large if σ is dominant over this μ u. So, in that case you will see that particle will retain its spherical shape when it travels through the channel. So, someone wants to know I subjected this particle inside this fluid channel and I subjected this droplet inside this fluid channel and the droplet will the droplet remain a droplet or droplet will split into several droplets think of it oil water I mean you have water flow and inside I have put an oil drop and then whether the oil drop will break into multiple small droplets or oil drop will retain its same size and shape as oil droplet and continue the flow. So, who decides that this capillary number will decide that. Similarly, when it comes to Weber number it is the inertia force by surface tension force.

Weber number, $W_e = \frac{\rho V^2 L}{\sigma}$

So, it is you have some unique situations. For example, in one situation you may have that you may have a case where a fluid is coming out of a channel of a capillary, and then any cylindrical string that comes out of a capillary it is one this cylindrical string will undergo an instability-induced breakup. You might have seen this as a dripping faucet problem. You open a tap very slowly, a cylindrical string comes out, and you see that the cylindrical string breaks down into small droplets. So, it cannot retain its cylindrical shape all through. Beyond a small distance, you will find that instability sets in and that instability will cause the formation of droplets and typically the formation of drops typically, the diameter of the drop is closely linked to the circumference of the faucet of that tap outlet.

So, these are the things that happen. Now, suppose I have a capillary from which a cylindrical string comes out. So, this is supposed to break down as per the instability. This instability goes by the name of a scientist it is called Rayleigh instability. So, because of this instability you will see that the liquid string will break down into drops or in this case droplets. Now, here, will the droplet form very near this? If this is the outlet of that nozzle, will the droplet start forming here itself, or will the droplet start forming there here in this all the way here?

So, it will maintain a cylindrical string, and then it will start forming droplets, or it could be that within a very short distance, it starts forming droplets. So, I mean, who decides that it is the Weber number because if you have too much inertia is provided with the fluid, the inertia will make sure that the fluid will continue as a string for quite some distance and then instability induced droplet breakup starts setting in. Whereas, when it comes to the Weber number with which, when it comes to low inertia fluid having low inertia, the droplet formation will happen at the tip itself. There are all these Froude numbers. It is important in the case of stirring; if you have a stirrer working there, you will have these numbers are so important. So, you have these all these numbers they have their unique significance.

So, you have to appreciate as you try to go into the physics of those problems you will see that these numbers had to be worked with. So, here in this case we have now when it there are similarities we have to talk about similarities there are three forms of similarities possible. Similarity in the sense that I build a model and I they in a real life there is there is a let us say I have a flow model here in front of me on the table and I have one that is implemented in a real scale which is much larger. So, then, when we say that here, what is happening is very same as what is going to happen there? For example, I measured Δp in this model I said we said a lot of things about experiments we will perform experiments and find out empirical relations etc.

So, here I perform some experiments. I get some information, and I link the dimensionless numbers. I found the exponents a, b, c, etc. So, that is I find an empirical correlation here. Let us say Euler number is some constant into Reynolds number to the power this etc. So, I have an empirical correlation and I say that this empirical correlation would be valid when it comes to a large scale real life prototype or real life plant process plant where this is operational. So, for that to extend this equation to there some similarity has to be established that is that they are probably the length of the pipe is much larger diameter of the pipe would be much larger because that is on a bigger scale.

And here I am working with a much smaller pipe where I when I perform these experiments. So, how they have to be similar? There are three forms of similarities possible: one is known as geometric similarity, another is kinematic similarity, and third one is dynamic similarity. Geometric means model and prototype have same shape linear dimensions on model and prototype. Kinematic similarity is velocity at corresponding points that means I am in the pipe one third distance away from the one third of the total radius away from the wall the velocity is same as the corresponding point in that the real life model the velocities have to be same. So, kinematic similarity velocity at corresponding points on a model and prototype differ only by they differ, but differ by a constant scale factor.

So, that scale factor has to be constant every at every point or every corresponding points. So, that is that is known as the kinematic similarity. Geometric similarity is the most simple thing I am working with a pipe of diameter this and length this their diameter is 10 times length is 10 times model and prototype has the same shape linear dimensions on model and prototype shape is same, but the dimensions change, but they are geometrically similar. You have the kinematic similarity where the corresponding points the velocity changes or pressure changes the other corresponding points they are velocity in particular when it comes to kinematic similarity it is the velocity. Velocity of corresponding points change by a constant scale factor.

That means, if that is 10 times whatever value I have in a model in hand and whatever value in the real prototype that I have. So, if this is 10 times that then I take another arbitrary point and the corresponding point I take there they all that also has to be 10 times the model value. So, all the corresponding points will have the kinematic similarity. That means, same scale factor will apply and the dynamic similarity which is most rigorous which is said the forces on model and prototype differ only by a constant scale factor.

For dynamic similarity

$$\left(\frac{\rho V D}{\mu}\right)_{model} = \left(\frac{\rho V D}{\mu}\right)_{prototype}$$

$$\left(\frac{F}{\rho V^2 D^2}\right)_{model} = \left(\frac{F}{\rho V^2 D^2}\right)_{prototype}$$

In other words I mean frankly speaking I do not understand this meaning of this word as such, but what I can tell you is when we talk about these let us say I come up for a drag force if I come up with this expression for drag force then we have to ensure when it comes to dynamic similarity the these dimensionless number will be same in model as well as in prototype and these dimensionless number will also be same as the model and the prototype. That means I have to here I run the experiment at the Reynolds number of 1000. There also, I have to run the actual flow take place at the Reynolds number 1000. So, this is known as dynamic similarity I am retaining forces on model the forces on model and prototype differ only by a constant scale factor. So, force may differ, but the entire net effect of dimensionless number that I have to retain the same dimensionless number. So, if the actual flow takes place at certain Reynolds number I have to implement that Reynolds number in my model study as well and then get those results.

So, that is most stringent and that is now referred as dynamic similarity, and when one goes for a scale up from the model in I mean suppose someone performs experiment and then that experiment they want to scale up. So, during scale up one has to ensure that the dimensionless numbers that are instrumental in arriving at those equations or you are relating one dimensionless number to the other and you want that relation to be valid in the prototype. So, in the prototype you have to have the same values of dimensionless numbers that you have studied here. Velocity may be different, density may be different, viscosity may be different. So, all together a different fluid, but dimensionless numbers have to match their value has to match then you can say this relation that we have I can extend it there.

So, that is known as dynamic similarity and typically that is something which we try to get. That is all as far as this lecture module is concerned. Thank you for your attention.