Momentum Transfer in Fluids
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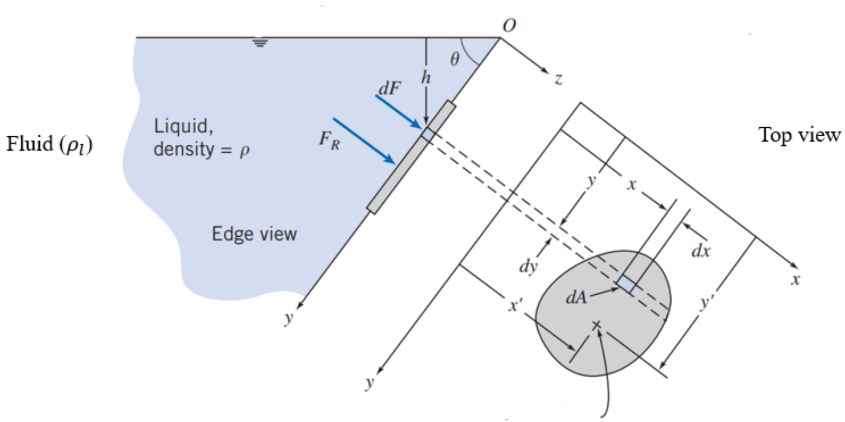
I welcome you to this lecture on momentum transfer in fluids. We have been discussing fluid statics. That is, we started with Newton's second law, and then we imposed that acceleration is 0, and you have the case where we are trying to balance between the pressure and pressure across various coordinates. So, there we have already discussed how to measure pressure, what are the uses of manometers and what all issues will come up if you have a compressible fluid that is density varying with height. Now, we are going to talk about this hydrostatic force on a submerged surface. So, that is something which we would like to discuss or we would like to delve into.



Now, here, in this case, what I have is a submerged surface, which is shown here. This is a so called submerged surface. This is the submerged surface. So, this submerged surface what all forces what would be the resultant force acting on this submerged surface that we are trying to figure out.

What we see here is that there would be because of this fluid height since this is a fluid and this is submerged. So, at this point let us say if I talk about this differential element here. This differential element of dimension this is the y axis. So, this is of dimension dy this dimension is dy and then it is this z this is the z axis. So, this dimension is dz and x is perpendicular to the screen.

Now, if we see the view from the third dimension it would be we are picking up this differential area dA whose dimension is dx and which is x distance away from the other edge of the submerged plate. So, we picked up a differential element that means if this is the submerged plate. So, we picked up a differential element here. So, this has dimensions of dx dy dz in that submerged plate and then we are trying to find out what all forces would be acting on this submerged plate. So, what we see here is that you have this is the height h and then you have what is the dF that is acting on this differential element and then what we need to do is we need to take care of all these dF's possible.

All these for every differential element there will be a force dF acting. So, we need to sum all these dF's and then we get the resultant force F_R. So, what we need to understand is what is this what is the value of F_R number 1 and number 2 what is the location through which this resultant force will act is it the halfway on the plate one third from the other one side or two third from the other side. So, what would be the location through which this resultant force will act if the plate is submerged in this form and lying in this form. So, what we see here is first of all this this element is subjected to hydrostatic pressure due to this height h.

So, obviously, this is h ρ g and so, as I move from this end of the plate to the other end these height increases for example, at this location these would be the height at some other location this would be the height like this. So, this would be the height and it will go like this. So, the dF varies as I move from this side to and continue to go in the y direction. So, as I move towards this so, the as I move on this plate the dF will vary. So, how do we find out what is the resultant force?

$$F_R = \int_A p\, dA$$

Resultant force is integration of P dA and the location at which this resultant force will act if I tell this coordinates are y' and x' then

$$y'F_R = \int_A y\, p\, dA, \qquad x'F_R = \int_A x\, p\, dA$$

First of all F r is equal to integral p dA what does it mean? Integration P dA. So, as far as this area dA is concerned the pressure force acting on this is p as far as next one is concerned next dA element is concerned there would be its unique pressure there would be unique pressure.

So, we need to sum all these pressures. So, integration p dA and then that would be if we sum them up that is summing in the sense that summing in terms of integral. So, that gives me the resultant force. So, that is how we ended up with this expression. How we end up with this expression? The point of application of the resultant force which is known as center of pressure that must be such that the momentum of resultant force about any axis is equal to the moment of distributed forces about the same axis.

That means, we talked about F_R which is the resultant force. The moment of this F_R with respect to some reference point has to be equal to sum of all these differential forces dF that are acting around that same point. So, what that means is if the position vector is r if the position vector from an arbitrary origin of coordinates to the point of application is designated as r and this r' e is the position vector as far as the resultant force is concerned. So, what we said is that moment of the distributed forces is equal to the moment of resultant force.

$$\vec{r'} \times \vec{F}_R = \int \vec{r} \times \vec{F} = -\int_A \vec{r} \times \vec{p}\ d\vec{A}$$

So, now, F is in turn P dA here we put a minus sign it is how you consider the resultant force whether it is acting on the plate or away from the plane mind it here. We have the z axis which is this way. So, any force acting in the negative z direction that would be considered as minus. So, here we put this as a minus sign r cross P dA this will be clear in a moment because I mean when we solve a problem on this here note this that this

$$d\vec{A} = -dA\,\hat{k}$$

because here the area vector dA the direction of dA is perpendicular to this. So, dA would be in this direction.

So, direction of area vector dA would be in this direction which is in negative z direction. So, that is why we are writing $d\vec{A} = -dA\,\hat{k}$. So, then when you put this minus $dA\,\hat{k}$ it would be this minus and this minus will cancel out. So, this would be a more rigorous description and the other thing is resultant force FR acts against the surface in a direction opposite to that of dA. The dA is acting in this direction and FR is acting in this direction.

So, that is they are working on opposite direction. Now, this x prime and y prime that has to be equal to this we need to find out this r prime that means, the location of this resultant force. So, location is given by x prime and y prime that means, x prime is from the edge because x is perpendicular to the screen. So, from one edge of the submerged plate as I move towards dA this area vector from one end of the submerged plate as I or x axis I put it here. So, this x axis this is the submerged plate as I move towards that dA.

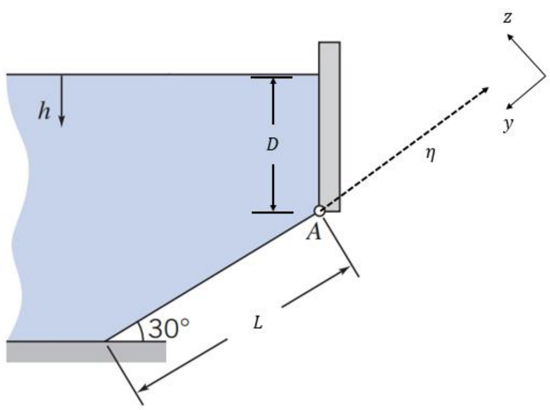
So, this is the x. So, now, we need to know what is the x prime and what is the y prime coordinates where this force is resultant force is acting. So, that is obtained by writing

$$(x'\vec{\imath} + y'\vec{\jmath}) \times F_R\,\hat{k} = \int (x\,\vec{\imath} + y\vec{\jmath}) \times d\vec{F} = -\int_A (x\,\vec{\imath} + y\vec{\jmath}) \times \vec{p}\;\,dA\,\hat{k}$$

$$-\hat{\jmath}x'F_R + \hat{\imath}y'F_R = \int_A (-\hat{\jmath}xp + \hat{\imath}yp)\,dA$$

$$y'F_R = \int_A y\,p\,dA\,, \qquad x'F_R = \int_A x\,p\,dA$$

So, so this gives me the location of this resultant location along which this result location at which the resultant force would be acting as far as this submerged plate is concerned. Let us look at first of all what is the implication of it. You must have seen log gates or you must have seen in process operations you must have heard of a concept called one way valve. The one way valve the objective is you might have seen what I am trying to say you might have seen something called a log gate or you might have seen something called a one way valve. The purpose is you can see it here in this picture.



I have a storage house I have a storage tank let us say and I have a plate here and this plate is hinged here to this wall and this plate is lying like this and it is having this water inside the tank. Now, when there is a flow in this direction so, then the hinge will allow this plate to rotate in this direction and then the fluid will enter into the tank. But when there is no flow when the flow is not there anymore then this plate will come and sit here and it will not allow any liquid to drain from the tank. So, essentially this is acting as a one way valve or this is acting as a reservoir. So, when there is a pressure extra pressure it will absorb that it will take the fluid in, but fluid will not be allowed to pass through, will not be allowed to release from here.

So, it is acting as a gate only one way the fluid will enter, but fluid will not be allowed to leave this place. It is not exactly the log gate it is I would say it is a storage device where a storage entity will allow only one way fluid to take place. So, ideally it is a one way valve that is perfectly acceptable. So, now, if you have a situation like this and if one is interested to know what would be the resultant force acting on this plate and at what location. Because if someone wants to support this at one point it may turn out that the height of this liquid level is too much and this plate is going to fail.

So, at where the resultant force will be acting, and if someone wants to support this where they should put that support so, that it will counter the resultant force. So, if someone wants to know that wants to perform that exercise. So, here this what we said just now that the equations that will work out very well. First of all the resultant force here mind it I have this is the y axis and this is the z axis. So, here z axis is taken in this direction.

If we take z axis that is fine earlier time we had taken z in the other direction and x axis is perpendicular to the screen. But since it is better to take this last time you may recall that what we have there we have started from here around point O we had taken this moment. So, we counted y from point O. So, here also we need to do the same thing here also we need to do we need to go all the way there and then from there we have to measure the η.

So, we have to measure the y. So, instead of instead of these we may we may like to write instead of y axis we can we can change this to η. So, we have a relation between y and η and that is $D + \eta\ \sin 30^o$. So, this is we are measuring y from this point. So, this is we are measuring y from this point and we say that we are measuring η from this point let us say η from this point. So, from this point we are measuring η.

So, we have a relation between y and η we have we will say that if this is η is measured from here and let us say I want to know what would be this what would be this height. So, this height would be on one hand this is D. So, up to this point it is D we see here. So, up to this point it is D and beyond this point this height since I measure η from here. So, this would be this angle is 30 degree.

So, that means, this angle is 30 degree. So, these are these height would be η sin 30 degree. So, what we are trying to say is that the height at any point any point the water level if we note this height would be the height of the water column would be D plus η sin 30 degree. So, we have to measure y from this point, y from this point, but it is it would be easier for us to measure η from this point. So, y and η they are simply related in the sense if we add whatever value of η we measure if we add the add what would be this length this length would be simply if this angle is 30 degree this angle is 30 degree that means, this angle is 30 degree.

If this angle is 30 degree this height is D. So, this would be this would be D sin 30 degree is D divided by this. So, sin 30 degree is d divided by this length let us say I call this capital H. So, then the sin 30 degree is equal to this. So, then this would be equal to D divided by sin 30 degree.

So, this length is simply D divided by sin 30 degree. So, you can see if I get a value of η and then you add D by sin 30 degree that will take you to y corresponding y value. So, this is so, with this now we work with now d y is equal to d η. So, that area $d\vec{A} = w\, d\eta\, \hat{k}$ this time we are not putting a minus sign because z is taken as in this direction itself and then we have instead of area you have w d η. So, now, we have now what is the pressure at any point? Pressure at any point is this is d this height is d plus η sin 30 degree.

So, this multiplied by ρ g that would be the pressure at any point. So, that is what we have written

$$p = \rho g\left(D + \eta\ \sin 30^o\right)$$

So, the resultant force that is acting would be integration p d A and then p is given as this and d A is given as w d η. What is this w d η? Area w is the width perpendicular to the screen. So, we are doing we are not working with any x value we are working.

So, our differential element that I choose has the dimension of d η on this direction and then perpendicular to the screen the entire width is considered entire width of the plate is considered which is not the dx. So, that is why we have w d η is the area I have picked up d η or d y it is the same thing. So, d η and w is the width perpendicular to the screen that is considered and that gives me the magnitude of the area d A. So, that is why we have this w d η and then we need to do the integration we end up with this as the resultant force. Next what we do is we try to find out what would be the line of action of resultant force.

$$\vec{F}_R = -\int_A p\, d\vec{A} = -\int_A \rho g(D + \eta\ \sin 30^o)\, w\, d\eta\, \hat{k}$$

$$\Rightarrow \vec{F}_R = \rho g w\left[D\eta + \frac{\eta^2}{2}\sin 30^\circ\right]_0^L \hat{k}$$

$$\Rightarrow \vec{F}_R = \rho g w\left[DL + \frac{L^2}{2}\sin 30^\circ\right]\hat{k}$$

We recognize that the line of action of resultant force must be such that the moment of the resultant force about any axis equal to the moment of distributed force about the same axis. So, we go by that same thing here. So,

$$F_R \eta' = \int_A \eta \, p \, dA$$

$$\Rightarrow \eta' = \frac{1}{F_R} \int_A \eta \, p \, dA$$

$$\Rightarrow \eta' = \frac{1}{F_R} \int_A \rho g (D + \eta \ \sin 30^o) \, w \, d\eta \, \hat{k}$$

$$\Rightarrow \eta' = \frac{\rho g w}{F_R} \left[\frac{D\eta^2}{2} + \frac{\eta^3}{3} \sin 30^\circ \right]$$

$$\Rightarrow \eta' = \frac{\rho g w}{F_R} \left[\frac{DL^2}{2} + \frac{L^3}{3} \sin 30^\circ \right]$$

So, now, this is the value of this is the value of η prime.

So, from η' if you want to convert this to y'. So, then you have to use this expression you put this η prime and y prime you will you can convert this toggle between η prime and y prime and you have this x prime. So, y prime we could find to find the x prime of course, you have to follow a very similar procedure though you do not have to one thing you may you may quickly note here that as far as the x axis is concerned which is perpendicular to the screen I mean we have the plate. So, this side it is this side it has the length L and perpendicular to the screen it has the width w.

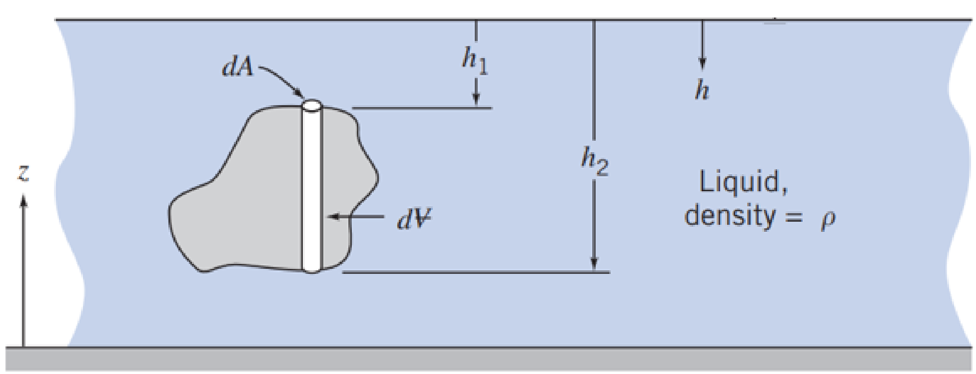
So, L into w is the total area of the plate. Now, we have picked up a differential element here on this side L and then we and we have rηined the entire W and that differential element that that gets generated. So, if we if this is the plate and we picked up this differential element and we have done we have seen how much of force is acting on this did the integration found out what is the F_R. And we found what should be these distance which is the η prime here. So, what is this η prime? We have we have seen this expression here.

So, this is the η prime. Now, when it comes to the x how from this edge where the this resultant force should be passing, resultant force would be passing η is the one axis this is the y axis I can understand or η axis I can understand. But here when it comes to x I do not have any reason to believe that along x direction there is any change it is all the same that is why I have taken the entire $w \, d\eta$. So, then it is invariably the resultant force would be acting if this if this is the this is the η prime and resultant force would be acting somewhere on this line. So, it would be simply halfway between the two.

So, if this is w so, it would be simply w by 2. So, it would be η distance away from this point and w by 2 distance from the from this side. So, that is that is what we we are we we will see this intuitively it makes sense that the it would since in the in the x direction there is no change involved. So, it would be just the center of that plate at which the F_R will act. So, you have we can we have a way to find out what is x prime and what is y prime. So, location of this resultant force is found out you can put the value of L F_R is already found out from here.

So, you can put those values and you can find out what would be the dimensions here and what is the resultant force acting. So, if someone wants to support this this someone wants to support this plate. So, that it does not bend or it does not rupture. So, the location through which the resultant force passes and the amount or the magnitude of the resultant force can be found out immediately. You will note here in this problem we did not talk about the atmospheric pressure.

The understanding is that here we have the atmospheric pressure here we have the atmospheric pressure and here on this side also there is atmospheric pressure acting. So, that is why we have not . I mean, if you want to consider atmospheric pressure, then you need to add atmospheric pressure on this side itself in the pressure term, and then you have to subtract atmospheric pressure from this side itself. So, net result would be the same. So, that is why you may see that we work only with the h ρ g term and do not go to the atmospheric pressure that is acting on top of the liquid because this is an open tank, and the pressure that is coming from the side. The other aspect that I quickly I need to touch upon which you have already studied in various occasions that is buoyancy and this buoyancy is very simply if you have an object dipped in the liquid.



So, then this object let us say it has a volume capital V struck through. Now, we take a differential area dA and let us say the top here is the water level is h_1 and the bottom the water level is h_2 I mean if we go to this reference the water level is h_2 and here it is h_1. So, the difference is h_2 minus h_1 this height is height difference is h_2 minus h_1. We pick up a differential element dA and see the volume of this. So, it would be h_2 minus h_1 into dA that gives me the volume of this differential element.

So, that is the volume of this differential element and then you see here that this differential element is subjected to a pressure and because of this water level it is having a h ρ g that is h_1 ρ g that is the amount of hydrostatic force acting on this surface whereas, the hydrostatic force acting on this surface is h_2 ρ g. So, this here it is h_2 ρ g. So, what is the difference between these forces? One is h_2 ρ g and here it is h_1 ρ g. So, this difference is this as far as this fluid element is concerned it is subjected to this difference in forces. So, that is why you can see ρ g into h_2 minus h_1 dA that is the resultant force acting upward here it is less force here it is more force.

So, that means, this object or part of this object this differential element the cylindrical element that I have picked up whose volume is dV. So, that is subjected to an upward force which is this force dFz ρ g into h_2 minus h_1 into dA. So, this upward force is acting on this differential element that has a volume h_2 minus h_1 into dA. So, net upward force would be in this case Fz would be equal to ρ g into V the volume of this. So, essentially that is now so, that is the upward force that is referred as the buoyancy force and this buoyancy force is so, buoyancy force here ρ is the density of the liquid ρ is the density of the liquid.

$$F_z = (p_0 + \rho g h_2)\, dA\ - (p_0 + \rho g h_1)\, dA$$

$$dF_z = \rho g (h_2 - h_1)\, dA$$

So, this is the on one hand you have the gravity force. So, you have F this is acting upward and the gravity force which is acting downward. So, this is this is this if I write as k hat the gravity force would be the ρ of the solid what you have into g into volume k hat and that would be in the minus sign. So, this is the gravity force acting downward this is the buoyancy force acting upward. So, all objects will be subjected to gravity plus buoyancy. We have very different cases where these gravity and buoyancy those would be coming and there we would be looking into their interplay.

I will be particularly these are important when it comes to settling when a particle is settling when a particle is settling the particle will be subjected to gravity force and at the same time there would be a buoyancy force upward. The only difference is one is ρ liquid and other is ρ solid and so, that or there is buoyancy corrected gravitational force that these are also these terms are common. So, during the settling of a particle, this buoyancy force comes into play when it comes to process engineering. We will discuss this. We will use of the term buoyancy in other contexts as well down the line in the flow and settling of solids.

So, this is however, this is a part of fluid statics definitely. So, this is all I have as far as the fluid statics is concerned. I have already talked about the various ways the pressure can be measured for example, use of manometer, use of differential pressure transducer, use of Bourdon gauge an operation of a Bourdon gauge is interesting you may like to review the operation of a Bourdon gauge because these gauge is so popular it is very cheap and everybody is using it. So, how it is how the internals work there. So, these already we have touched upon and we will continue working, but fluid as far as the fluid statics is concerned I think formal lecture ends here. We will discuss about other aspects and there as and when fluid statics comes in we will take it up.

That is all I have as far as this lecture module is concerned. Thank you very much for your attention.