Momentum Transfer in Fluids
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I welcome you to this lecture on momentum transfer in fluids. In this lecture, I am going to talk about fluid statics and some associated problems. However, before I proceed further on fluid statics, I have already introduced what fluid statics would be. You may remember when we talked about Newton's second law and application of Newton's second law to reach Euler equation etc. So, there in this context I said there are 3 special cases. In one case, the acceleration is 0, another case acceleration would be Ax Ay Az where the liquid is hauled with an acceleration, the entire liquid and mass is hauled at an acceleration and the third case was that there is a fluid flow that would be a acceleration is given by the substantial derivative of the velocity.

Now, the first case that means when acceleration is 0 that is essentially fluid static. So, we have some idea what fluid statics is already. Now, before we proceed on fluid statics, one aspect of our previous understanding of this inviscid flow is that we talked about this Euler equation that I said with acceleration given by the substantial derivative of velocity that from that Euler equation we can convert that to the energy as well. We have not discussed; we have only talked about the force balance and Newton applying Newton's second law.

We can apply, we can extend that by conservation of energy as well, and, so I will do this in a quick one or two slides, I will just quickly mention how that is done and what are the implications of it before I proceed on traditional fluid statics. So, what we see here is that for irrotational flow we have said that $-\nabla p + \rho \vec{g}$ this was that Newton's second law and then we when we have the acceleration term is given by the substantial derivative we call that Euler equation. So, this is what it is the basic equation that we get from Newton's second law.

$$\rho \vec{a} = -\nabla p + \rho \vec{g}$$

Here we have further we can say that this acceleration is given by dv dt the substantial derivative and then in turn we can write this substantial derivative is essentially the local acceleration and the convective acceleration. And then there is a vector identity by which one can show that this convective acceleration can be written in this form where ζ is the vorticity we have we have already discussed vorticity.

$$\rho\vec{a} = \rho\frac{D\vec{V}}{Dt}$$

$$\rho\frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$$

$$\Rightarrow \rho\frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right) + \zeta \times V$$

So, this is one thing that can be shown and then then if we take so if we bring in there if we bring into this Euler equation instead of $\rho\vec{a}$ we put this that is instead of. So, instead of $\rho\vec{a}$ we put this if we put this as $\rho\vec{a}$. So, what we get is this expression. So, if we divide both sides by ρ if this equation divide both sides by ρ we end up with this expression. And then we note here that the left hand side represents force per mass.

Dividing both sides of Euler equation by ρ

$$\frac{\partial\vec{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right) + \zeta \times V + \frac{\nabla p}{\rho} - \vec{g} = 0$$

So, if it is force per mass then if we try if we take a dot if we take a dot product of the left hand side with an arbitrary displacement vector dr that means whatever is the left side left hand side if I take the left hand side this is what is the left hand side we have dot dr that will also remain as 0 and this dot dr this gives me the energy because force into displacement that is the energy.

L.H.S. represent (Force/mass)

A dot product of LHS with an arbitrary displacement vector $\overrightarrow{dr}$ gives work done or energy.

$$\Rightarrow \left[\frac{\partial\vec{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right) + \zeta \times V + \frac{\nabla p}{\rho} - \vec{g}\right]\cdot dr = 0$$

So, if I take a dot product that gives me 0 and then if we if we take the so what all terms are there we have one this $\zeta \times V$ and we have these individual terms. So, for the time being if we stop thinking about this $\zeta \times V$ and only focus on other terms I will come back to this $\zeta \times V$ later. If I focus on other terms then what we end up with is we see that the left hand side becomes this quantity.

Without $(\zeta \times V).dr$ term, L.H.S. becomes

$$\frac{\partial V}{\partial t} \cdot dr + d\left(\frac{V^2}{2}\right) + \frac{dp}{\rho} + gdz = 0$$

$$\int_1^2 \frac{\partial v}{\partial t}. ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

$$\overrightarrow{dr} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

$$\Rightarrow \nabla p. \overrightarrow{dr} = \frac{\partial p}{\partial x}. dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz = dp$$

Similarly $\nabla\left(\frac{V^2}{2}\right). \overrightarrow{dr} = d\left(\frac{V^2}{2}\right)$

$$\vec{g} = -g\hat{k}$$

$$\Rightarrow \vec{g}. \overrightarrow{dr} = -g\hat{k} \cdot \left(\overrightarrow{dr} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\right)$$

$$\Rightarrow \vec{g}. \overrightarrow{dr} = -gdz$$

So,

$$(\zeta \times \mathrm{V}). \mathrm{dr} = 0$$

1. V=0; No flow (Hydrostatics/ Fluid statics)
2. $\zeta = 0$; irrotational flow
3. dr $\parallel$ V; Integration along a streamline
4. dr $\perp$ $(\zeta \times \mathrm{V})$; special and rare case, No need to consider

Bernoulli’s equation along a streamline

$$\int_1^2 \frac{\partial \mathrm{v}}{\partial \mathrm{t}}. \mathrm{ds} + \int_1^2 \frac{\mathrm{dp}}{\rho} + \frac{1}{2}(\mathrm{V}_2^2 - \mathrm{V}_1^2) + \mathrm{g}(\mathrm{z}_2 - \mathrm{z}_1) = 0$$

Gives energy conservation between points 1 and 2 along a streamline, where ds is the arc length.

For incompressible and steady flow,

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

For irrotational flow, integration can be performed between any two points on the flow field, and the constant will not vary.

Now, this integration if this integration is done that means, dr is chosen dr is parallel to v that means, integration is along a streamline and the other thing is dr is perpendicular to this is a very special case we are not considering at this time.

So, because of these reasons one can have $(\zeta \times V).dr = 0$. Now, if $(\zeta \times V).dr = 0$ we have seen we have seen that the integration that we mentioned now this expression is same as what we have here. So, now, this expression is essentially Bernoulli's equation if you have probably seen this Bernoulli's equation in the form of steady state. So, then this term would be absent because this is the unsteady state term and then if you have incompressible flow then in that case your ρ in this case would be constant. So, ρ will come out so, then it would be Δp by ρ.

So, that is why or in other words p 2 minus p 1 by ρ. So, that is why we said from here we get $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$. So, that is essentially what is Bernoulli's equation. Now, what all we are trying to so, essentially what we did we started with Euler equation what is Euler equation Newton's second law acceleration is only acceleration term is not nonzero acceleration it is not an in mass acceleration it is the substantial derivative of velocity it has both local and convective acceleration. And then we took that and then found we found that each term if I divide by ρ it is force per mass we take a dot product with the displacement dr and that has to be so, everything put together left hand side was 0.

So, dot dr will be equal to 0. So, then from there we simply end up in this equation we end up in this equation provided this $(\zeta \times V).dr = 0$. So, we arrive at Bernoulli's equation we arrive at Bernoulli's equation under special conditions one is fluid statics. Fluid statics means the velocity will be gone you show it will be only p by ρ and gz term would be coming or it flow has to be irrotational that is one of the primary concern. So, Bernoulli's equation can be applied only for inviscid flow though Bernoulli's equation has been extended to frictional flow or viscous flow, but that is some ad hoc arrangements have been done to put some additional term and we want to we want to you know make do with these, but rigorously Bernoulli's equation is meant for irrotational flow.

That means, curl of the velocity field has to be equal to 0 and I so, either that has to be satisfied for irrotational flow or you can apply Bernoulli's equation between two points as long as those two points belong to the same streamline. That means, dr is parallel to v. So, that is that is also another condition by which the Bernoulli's equation is satisfied. So, that is why you will find Bernoulli's equation can be applied along a streamline. You do not have to apply Bernoulli's equation for the entire cross section of the π.

You can you can apply Bernoulli's equation along a streamline as long as you track that. So, that is that is one issue we have to keep in mind very clearly as we proceed further. So, anyway this Euler equation we said now if we go to the we remember we had already arrived at that expression of $-\nabla p + \rho\vec{g}$ is equal to 0 because this acceleration term was 0 and that we said was fluid statics. My next I mean what I am going to do over next couple of lectures is we try to we try to dwell on this particular part and where how we arrived at this I have given you that again that surface force on a differential volume and the body force on that differential volume and how we arrived at these. You may like to put together those again I mean revise those go through them and see that these are I mean you recaπtulate what all we have considered while applying this Newton's second law.

But essentially those same equations are valid and I can probably proceed with this expression. I assume that you since we have worked on this before so, you can go through these go through these equations and figure that out how we arrived at this expression earlier in previous lecture. So, what we did at that time was that we found Δ p Δ x is equal to 0 Δ p Δ y is equal to 0 and minus Δ p Δ z is equal to ρ g. So, that means pressure does not I mean if I have a pool of fluid liquid. So, pressure does not let us say it is a tank rectangular tank and it is half filled with a liquid pressure does not if I πck up a point if I go in x direction if I go in y direction pressure does not change only if I go in z direction pressure will change.

So, that is what this says and this dp dz is equal to minus ρ g. Now, here in this case in this context we had talked about fluids in rigid body motion. So, that is that was also that we discussed earlier. Now, dp we said that dp dz is equal to minus ρ g and then we said that when ρ is constant ρ is not varying with z then we can do that dp this integration would be simply minus ρ g integration dz. Now, it is you are doing from 0.

1 to 2 and this I am doing from 0.1 to 2. So, this means p2 minus p1 is equal to minus ρ g z2 minus z1. How do you define let us say I am talking about as I go up this is my z direction. So, as I go up let us say I have these as 0.

1 and these as 0.2. So, p2 minus p1 is equal to z2 minus z1. So, z2 minus z1 here is this term is positive in this case this term is positive and p2 minus p1 then has to be equal to negative. That means, as I go up pressure decreases that makes perfect sense in

atmosphere as I go up the air pressure decreases. So, that is why we as I travel up the mountain people have breathing difficulties and as I go down if we are in a swimming pool and as I go down I is hard because the water pressure applies on our eyes. So, you can write Δ p is equal to h ρ g this we have seen from our earlier days from our even in our class understanding in class 11 12.

So, Δ p is equal to h ρ g that is essentially p2 minus p1 is ρ g into z2 minus z1. This concept is used to measure pressure between two points and a common one common instrument that is used is so called manometer. You must have seen these manometers in the laboratories in many different places. In fact, nowadays the blood pressure measurement is mostly digital. However, over last decade you see those gauges those gauges those are referred as Bourdon gauge.

These gauges there would be a gauge like this there will be an indicator and then indicator will move. There is a flexible tube which is being as the air goes in or as the pressure increases. There is some movement in that tube and so because of that the pointer moves. So, so this is Boudon gauge is also used to measure blood pressure you might you might have seen doctors using it. It does not measure the blood pressure. As such, they are measuring the pressure, and then there are some other theories attached to it by which the blood pressure is measured.

So, blood pressure is estimated. So, you have these, the Boudon gauge is used, but otherwise, you will see still some of the places you find blood pressure is still measured where this manometer is there, and manometer is used in many industrial contexts as well. Though most in these days in industries there is again digital meters are more common. So, one use instead of these manometers what is commonly used is what people use is differential pressure transducer where you have a diaphragm, and then you have a diaphragm, and you put this side there is more pressure this side there is less pressure. So, there would be a deformation in this diaphragm, and because of this, there would be certain changes in the electrical properties, which would be captured by applying certain voltage and measuring the current, etc. So, so this differential pressure transducer good part of it is it is you get a digital information of pressure right away.

Though the manometer here you give it gives you a visual and manometer is something which is very common, I mean still there are you will find in a lot of places manometers are used. So, and it is very inexpensive compared to differential pressure transducer definitely. So, if what you have here is something called a manometric fluid. So, this side let us say I have a pressure here and this side I have a pressure here.

So, let us say I have I call these the 0.1 and 0.2 this is 0.1 and this is 0.2. So, I am interested to know what is P 1 minus P 2.

So, I assume that the fluid that is there 0.1 and fluid that is there 0.2 these are of same fluid. Let us say this is this is known as fluid 2 or you can call these manometric fluid manometric fluid and you have the you have the fluid 1 and fluid 2. So, if someone which side the pressure is more I can see very well that this side this is depressed.

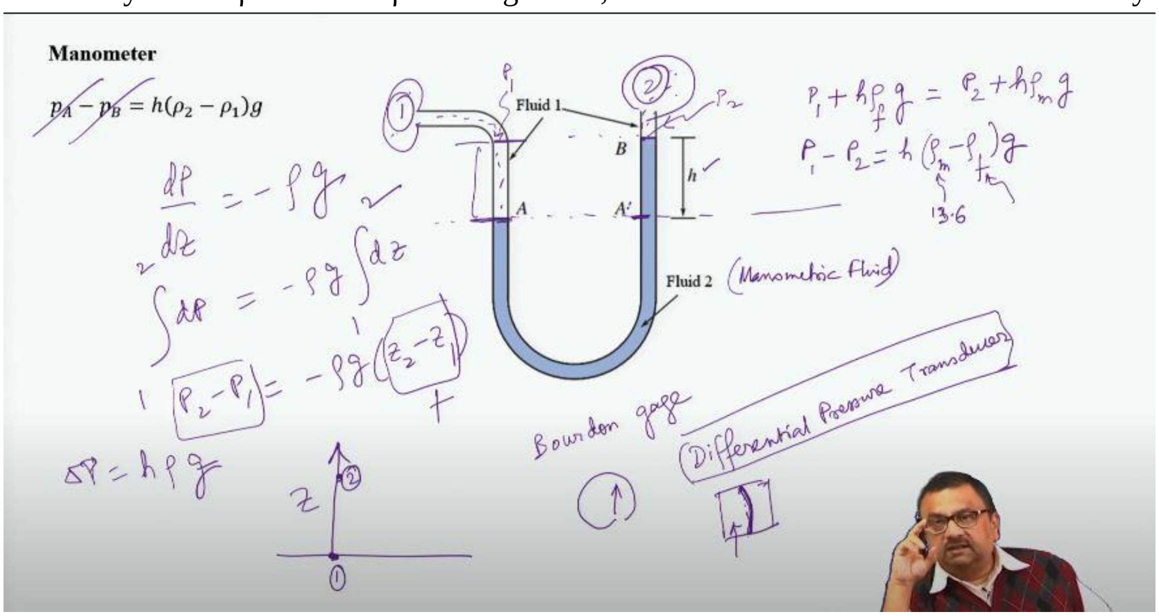
So, that means, that means, you have you have the since the manometric liquid density of the manometric liquid is more. So, you can expect that this side is at higher pressure. How? I will if I call if I take this as the datum. So, then I have if I consider this height to be this height to be h yeah that is that is what is given. So, what I see here is if I call this point this point the pressure is P 1 and this point the pressure is P 2.

So, then we will write P 1 plus P 1 plus this height what is h this fluid let us say I call this ρ f the fluid that we are whose pressure we are measuring into g. So, that gives me the pressure here at A and pressure at A prime I will get here that would be P 2 plus h the manometric fluid I call this ρ m into g. Now, these two are at the same level. So, pressure along the x direction does not change.

So, and it is the same fluid continuing. So, this has to be equal to this. So, what that means, is P 1 minus P 2 or P A minus P B they have given a P A minus P B is not done on the right statement I will call it P 1 minus P 2 that is

$$p_1 - p_2 = h(\rho_2 - \rho_1)g$$

The common conception is that it would be h into ρ into g ρ is the density of the manometric fluid, but that is not correct. You can see here that P 1 minus P 2 is essentially h into ρ m minus ρ f into g. Now, it common manometric fluid is mercury.



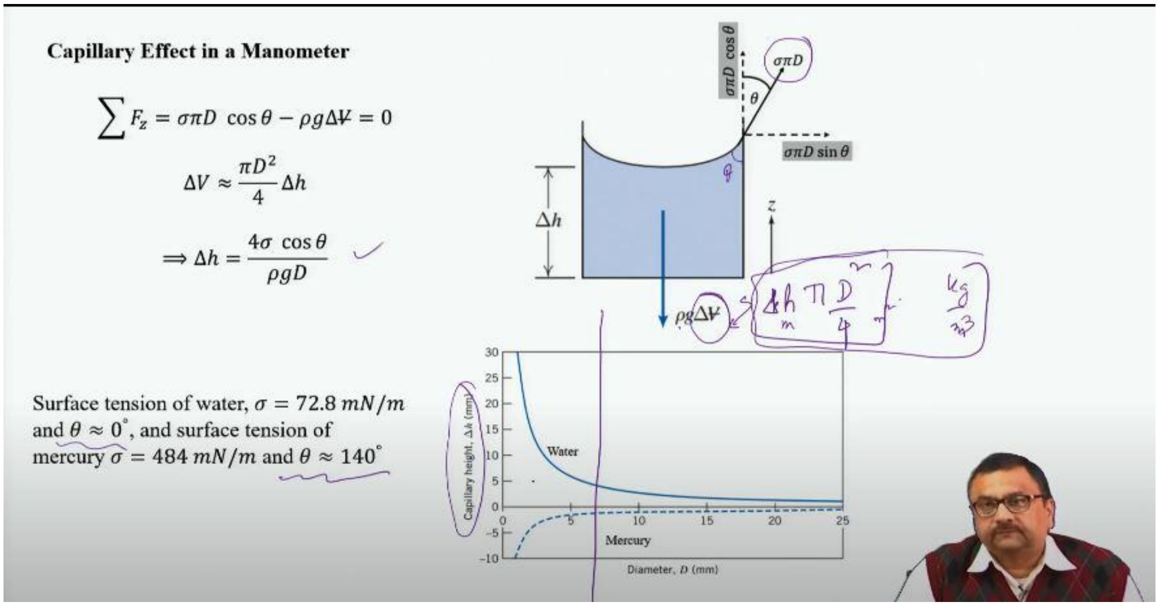
Mercury has a density of 13.6 gram per cc and common fluid that that you see water which is 1 gram per cc or it could be oil density would be 0.7 0.8 gram per cc. So, you if you use manometric fluid as mercury.

So, this value will take as 13.6 and this ρ f would be in case of water it would be 1 if it is oil it is 0.8 maybe. So, that has to be that density has to be subtracted that is the correct way of doing it and if you do not do it you get I mean get an approximate value P 1 minus P 2. So, this is something which you need to keep in mind. A couple of other things: number 1 when you are when you want to measure a small pressure drop, you prefer to have a manometric fluid whose density is low. I mean, you can go to oil as a manometric fluid, and it will be 0.8 because the small pressure, but you want a large deflection. If your deflection is on the order of let us say 1 centimeter and if your deflection is on the order of 10 centimeter to measure 1 centimeter by I estimation and to measure 10 centimeter. The error that you have in 1 centimeter let us say your error is let us say 2 millimeter. So, so this is it is so you have to measure 1 centimeter actually it could be 0.

8 centimeter or 1.2 centimeter and for 10 centimeter the error would be again 9.8 centimeter or 10.2 centimeter. So, so that is so that that means the amount of error is it would be much less if you have a larger height to work with I mean that the percentage error would be much less if you have larger height to work with. So, so that is why if you use a manometer you prefer to have if you want to measure small pressure drop you prefer to have a manometric liquid whose density is less.

So, that that would reflect the density is less means this term within bracket would be less. So, that means for the same pressure drop the h has to be more that means you will see a larger height difference and there your percentage error when you do the I estimation that would be less. So, that is that is one point you need to you need to be careful with and then there are couple of concerns that mean manometric fluid has to be clearly immiscible with the process fluid and it has to have some distinct color so that they can be separated and they should not evaporate easily or react easily with the process fluid. So, so those there are it has to be sufficiently inert. So, this there are certain considerations which you one must employ for manometric fluids.

In many places you will see manometers in industrial settings as well as in laboratories. Now, there is one point you must make note of is that what should be the diameter of this manometric tube. If the diameter is too small then there would be capillary rise will come into play capillary rise is the rise arising due to surface tension forces. So, you can see here that any liquid in a tube will be pulled say for example, I have it depends on depending on the contact angle for example, I have water and I have glass and I can see that the glass water weights glass contact angle is small.



So, water will be pulled in the glass capillary. So, you will find there is a capillary rise. So, that capillary rise will be mess with the actual measurements. So, here we can see that what would be the I mean the surface tension force would be σ is Newton per meter multiplied by the perimeter of this which is π D. So, $\sigma \pi D$ would be the Newton per meter into meter. So, that is the force by which the liquid many squares will be pulled and then the vertical component would be $\sigma \pi D$ into cos θ since this angle is θ and then you have you have this because of this pull the liquid will rise and if the rise is Δh then you have the corresponding ΔV is Δh that you have into π D square by 4 π D square by 4 is the area that multiplied by ΔH. So, this gives me meter into meter square. So, that gives me meter cube multiplied by ρ g. So, that gives you ρ into this is this is meter cube ρ gives me kg per meter cube.

So, then this meter cube and this meter cube will cancel out. So, this gives me so many kg multiplied by g. So, m into g that would be the downward force because of gravity. So, these two can be equated and one can find out what is the Δh. So, if somebody plots this Δh capillary height Δh as a function of diameter. As the diameter increases you can see Δh is inversely proportional to D it is obvious when capillary rise would be more when the capillary is small the diameter is small capillary rise does not affect the problem when you have diameter sufficiently large.

So, one can plot Δh as a function of the diameter D and you can see that the capillary height decreases for glass water system and capillary height increases for mercury glass system. So, this is because mercury the contact angle is reversed we discussed this in one of the previous lecture. So, you can see because here it is θ equal to $140^0$for mercury glass system and θ equal to 0 degree near 0^0 for glass water system. So, because of this so, this plot tells me that any manometer tube that you work with if you if your diameter of the tube is something of this or that means, it is greater than 6 millimeter 7 millimeter then one should be happy with the accuracy. If it is less than 6 millimeter or less than 5 millimeter there could be the capillary rise can affect the affect the measurements.

This is all as far as this present lecture module is concerned, but I will continue this fluid lecture on fluid statics as a part of next lecture. Thank you very much for your attention.