Momentum Transfer in Fluids

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Week-07
Lecture-35

I welcome you to this lecture on Momentum Transfer in Fluids. We have been discussing complex potential. We have been trying to understand a superposition of different elemental flows and how the use of complex potential helps us in understanding this superposition. So, what we are going to do today is we are trying to first see the superposition. Superposition in the sense we are going to have superposition of uniform flow and a source.

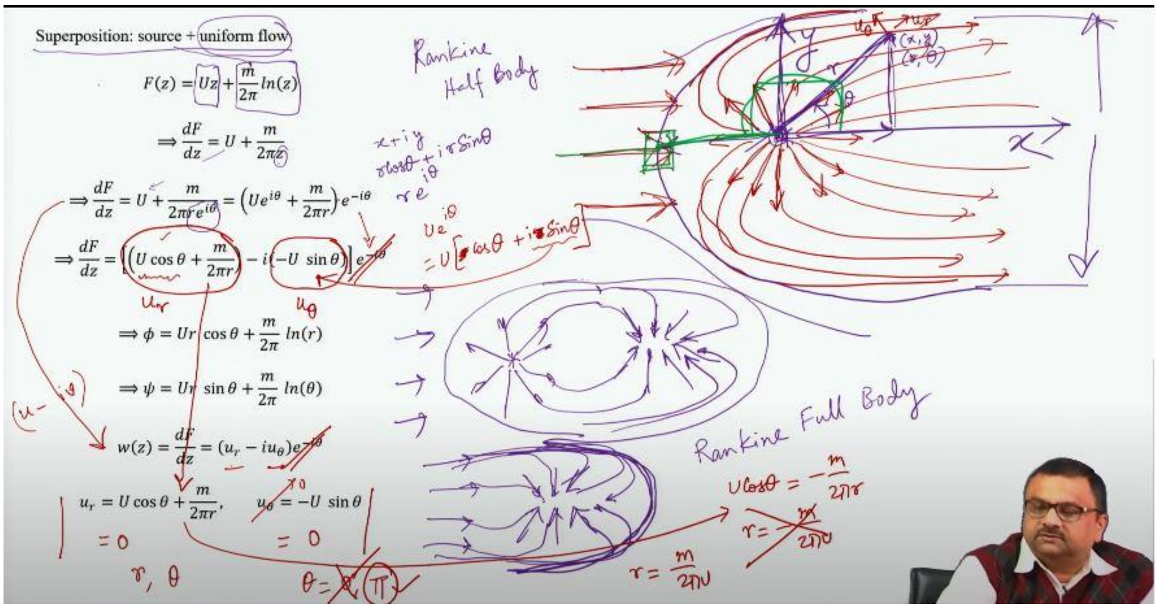
So, we have uniform flow and a source and if we have the combination of these, how the streamlines look like. Intuitively, if we do not go by the theories and if we try to trace the streamlines, I can expect that these streamlines will go like this. These would be the streamlines. If we try to see further downstream, these would be the streamlines.

What will happen with these? Because these cannot continue because we have superposed uniform flow. So, because of this uniform flow, we will see that these streamlines will start bending and eventually these streamline will also bend like this. The same is the next one. Streamlines cannot touch each other, but they will go like this. So, as if I have an envelope created because of these streamlines, that is what we can see here.

Some envelope is getting created, and if someone wants to know that where all these effect of this source will be felt, so this envelope will remain within this envelope. I mean I can give you an example. For example, in a river, you have uniform flow going in, and you have a point source. That means you have a pipe put there, you need depth, you put it in that stream and start pushing some fluid out. Maybe some waste stream, maybe some excess water you are putting there.

So, now you are trying to find out where all that fluid will travel, where all that fluid will go. So, we can see by looking at these streamlines is that we can see that there would be an envelope within which the effect of this source will be felt. For example, the fluid that you are putting in may have certain properties that may cause some harm, that may cause some improvement, an enhancement to certain properties of the fluid that is there, that is flowing. So, up to what distance the effect of the source will be felt, and up to what distance the fluid that is being pushed through the source will be traveling? So, this is the envelope I can think of.

I mean intuitively if you do not go by this, I mean if you do not go to the rigorous theories here. The envelope that is being created it has a name to it. The name given to this is Rankine half body and you can see that again intuitively if you have a source and a sink and a uniform flow. So, let us say I have a source here. So, this is what it is the source and I have a sink here.



So, fluid is being drawn toward the sink, and on top of that, I have a uniform flow. So, here I would be forming a Rankine half body, a shape of a Rankine half body, but here these streamlines that are coming out, they will be now drawn towards the sink. So, you will find that the envelope that you create this is the form of envelope that that shape would come up because here also the fluid that is going in fluid would be drawn here. So, here also on this side there will be another half of the envelope coming in. This is referred to as Rankine full body.

And this type of exercise is commonly done to simulate the flow around an airfoil. So, from the concept from the point of view of aerodynamics, so an airfoil that can be simulated by considering a source and sink, only source and uniform flow. So, this is intuitively we can have this. In fact, if I say that instead of a source, had there been a sink, then what we would have seen is there is only uniform flow and a sink. So, sink is drawing fluid.

So, what you will see in that case is that the uniform flow is going in here, uniform flow is going in here. So, you will find that you will produce a structure which is the effect of the sink will be felt within this envelope. So, now, if we try to find out what are the dimensions of this envelope, so that we can characterize it better, I mean where all fluids are traveling, where all it is going to impact. So, we have to take you can see here superposition of source + uniform flow. Now, for the elemental flow of uniform flow, we have the complex potential given as U z, we had discussed this in previous lecture.

And for source, we have these $\frac{m}{2\pi} ln(z)$, where m is the strength of the source. So, that means, meter cube per second per meter perpendicular to the screen, m is meter cube per second per meter perpendicular to the screen and z is x + iy the complex potential where is x and y, this is the y axis, this is the y axis and this is the x axis. So, this is x and this is y and z is simply x + iy in this case. So, now, suppose I want to know someone wants to know what is how far on the upstream side the effect of the source will be failed that means, what is this distance? What is this distance? And if I proceed all the way to infinity in this case, what is this distance? So, that more or less gives me a fair idea how the envelope will look like. So, I superpose this u z + $\frac{m}{2\pi} ln(z)$, the only thing is now I have to after summing them up, I have to go with real part and the complex part and then we have to differentiate it with respect to z to find out what is u and v and probably we will work with u r and u θ and we already have those vector identities to toggle between the Cartesian u and v and the cylindrical u r and u θ.

$$F(z) = Uz + \frac{m}{2\pi} ln(z)$$

$$\Rightarrow \frac{dF}{dz} = U + \frac{m}{2\pi z}$$

So, if I take the derivative with respect to z what we have is the first term u z gives me u and m by 2 π ln z is $\frac{m}{2\pi}$. So, because derivative of ln z with respect to z is 1 by z. So, then this is we write u as it is and m by 2 π z this z is replaced by r e to the power i θ you may recall we have already done this several times z is written as x + i y and x is r cos θ and y is r sin θ as per this. We can have x and y here or we can have let us say a point here I am saying this is this as coordinate x and y or I can say this point has the coordinate r and θ. So, I can write it in terms of r θ as well.

$$\frac{dF}{dz} = U + \frac{m}{2\pi r e^{i\theta}} = \left(Ue^{i\theta} + \frac{m}{2\pi r}\right) e^{-i\theta}$$

So, what would be in that case the x? x is this. So, x is this distance. So, this x is r cos θ and this y is r sin θ. So, instead of x I write r cos θ, y I write r sin θ and using Euler theorem we write r common r take r outside cos θ + i sin θ using Euler theorem it would be e to the power i θ. So, that is exactly we have written instead of this z we have written r e to the power i θ.

Then we have this I have taken this e to the power i θ goes to the numerator and becomes e to the power - i θ and then e to the power - i θ I take common that is outside this envelope. So, we have to multiply here e to the power i θ. So, that this and this they cancel out and you are left with u only. So, it would be u to the power i θ + this m by 2 π r that remains as it is e to the power - i θ outside. You may recall that finally, we are going to equate this with what we are going to equate this with u r - i u θ e to the power - i θ etcetera and there we will take this e to the power - i θ we will cancel out and all those.

So, there is an advantage of keeping this e to the power - i θ outside. Let us say we have now u e to the power i θsuppose I mean forget about this that we have used this several times this we have used several times let us see what treatment we can do here in this case. Here in this case we have u e to the power i θ this e to the power - i θ remains as it is e to the power - i θ u e to the power i θ. Now, if we break this up because e to the power i θ I cannot split them into real and imaginary. So, u e to the so, for that purpose what we do is u e to the power i θ I split it.

$$\frac{dF}{dz} = \left[\left(U\cos\theta + \frac{m}{2\pi r}\right) - i\left(-U\ \sin\theta\right)\right]e^{-i\theta}$$

So, what I write here is u e to the power i θ is written as r cos θ + i r sin θ and then this if I take the real part and leave out the imaginary part. So, u r cos θ that comes here sorry here since it is only e to the power i θ r should not be there is no r here no r u cos θ + i sin θ. So, now if I take only the real part it is u cos θ and if the imaginary part is appearing as - i - u sin θ that is + i u sin θ. So, imaginary part is here real part is u cos θ is here and m by 2 π r goes with the real part. So, I put this within the bracket u cos θ + m by 2 π r.

So, now, I can link this you can see this $\frac{dF}{dz}$I have already said that $\frac{dF}{dz}$ is equal to u - iv and u - iv by vector identity we have shown several times that I u is in x direction v is in y direction and instead of u and v we can work with something called u r and u θ u r and u θ. So, that is possible. So, and there is a vector identity by which we can link u - iv to u r - i u θ e to the power - i θ. So, that is a vector identity u - iv is this and that vector identity we have shown when we defined earlier when we defined potential function as in previous lecture in the context of complex potential and stream function potential function. So, this you can utilize this vector identity and we can see here that this e to the power - i θ and this e to the power - i θ they will cancel out this e to the power - i θ here $\frac{dF}{dz}$ in this expression and this e to the power - i θ dF dz in this expression they will cancel out.

$$\Rightarrow \phi = Ur\ \cos\theta + \frac{m}{2\pi}\ ln(r)$$

$$\Longrightarrow \psi = Ur\ \sin\theta + \frac{m}{2\pi}\ ln(\theta)$$

$$w(z) = \frac{dF}{dz} = (u_r - iu_\theta)e^{-i\theta}$$

$$u_r = U\cos\theta + \frac{m}{2\pi r},\ \ u_\theta = -U\ \sin\theta$$

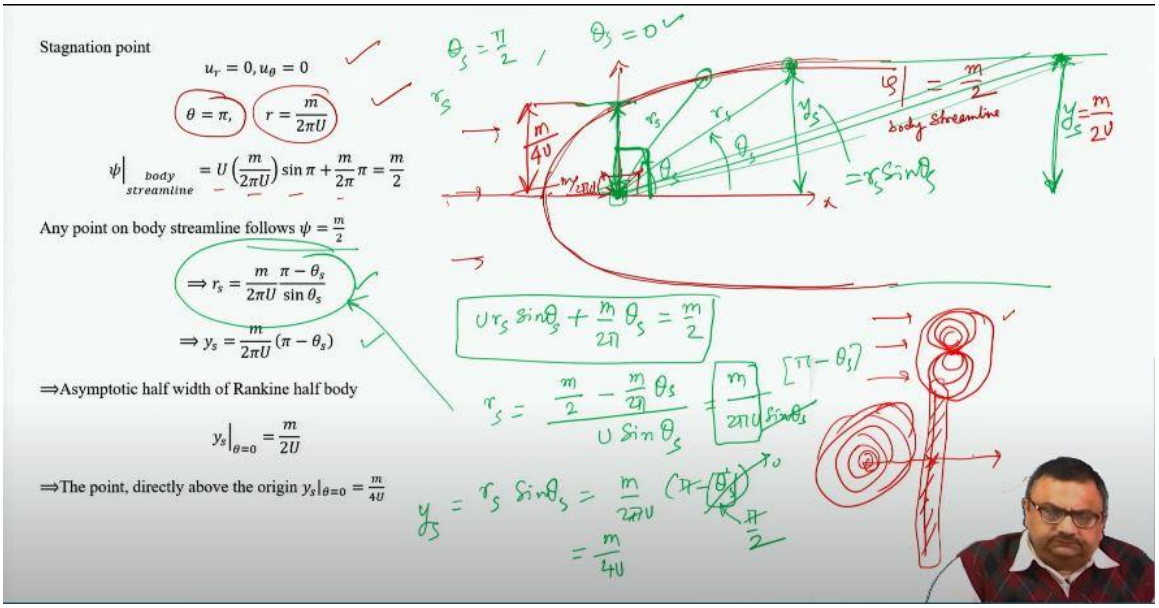
So, I can write or in other words I mean even not that I mean I should not be cancelling with this here rather I would be cancelling it here I would be cancelling it here this e to the power - i θ this e to the power - i θ is cancelling with this e to the power - i θ. So, what we are left with is this is the real component and this real component I will call U r - i u θ right u r - i u θ. So, i u θso, this term would be now u θ. So, if I put them together U r is u cos θ. So, we see here that $u_r = U\cos\theta + \frac{m}{2\pi r}$ and $u_\theta = -U\ \sin\theta$.

So, now, if I look at if I try to find out what is going on at this point I can see here that one streamline is going from the left hand side one streamline is going from the left hand side and another streamline from the source is coming from right hand side. So, at this point they merge and I expect that one streamline and other streamline they are coming from the opposite directions this is going to be a stagnation point. That means, the velocity has to be equal to 0 one streamline is countering the other streamline. So this is known as the stagnation point. So, if we try to find out where this stagnation point located on this envelope or in this picture.

So, for that we have to equate u r as equal to 0 and u θ as equal to 0. So, if we set u r is equal to 0 we get u cos θ is equal to. So, what are you trying to find out if I set u r as 0 and u θ as 0. So, I will come up with some r and some θ. So, that r and that θ will show me the coordinates of that stagnation point.

So, we can see here that if we put u r as equal to 0 then this becomes u r as equal to 0 takes me to u cos θ is equal to - m by 2 π r and u θ is equal to - u sin θso when this is equal to 0 when we set u θ is equal to 0 u cannot be equal to 0. So, that means, sin θ is equal to 0. Sin θ is equal to 0 means θ will take the value of either 0 or π these are the 2 values possible with θ for sin θ equal to 0. If we take θ is equal to 0 so then cos θ is equal to 1 in that case. So, cos θ is equal to 1 means then if I try to find out the value of r because θ I already have either 0 or π with θ as 0 r would be equal to in that case - $\frac{m}{2\pi U}$ and I do not see how r can be negative in this context I mean I cannot physically conceptualize.

So, that is why this we are not proceeding instead I take θ. So, θ equal to 0, I am putting this out θ is equal to π if I take cos θ in that case would be - 1 and then we can have r as $\frac{m}{2\pi U}$. So, that what that means is that θ is equal to π that means, θ is equal to π means these angle should this is the θ, θ is now π. So, that means, it would be on this line somewhere and r would be equal to $\frac{m}{2\pi U}$. So, that means, this distance would be equal to $\frac{m}{2\pi U}$.



So, this is what this exercise tells me. So, we have the stagnation point is given θ equal to π this is the feasible solution we have already stated. Now, if I try to find out that we said that this is the rank in half body. So, we have this is the origin, this is x, this is y, this is the source, this is the source and this is the uniform flow. And then we said that this is the plot here and we are trying to find out what is the this streamline the streamline that is passing through the stagnation point this streamline is referred as body streamline.

So, what would be the ψ value for the body streamline? If that is what we try to find out next so, that we can find out the other parameters as well. So, the stream function for body streamline would be what is the equation for stream function? Stream function in this case f z is equal to U z + m by 2 π ln z when we did that time if we do not do the derivative of this if this f z is equal to on the other hand ϕ + i ψ and then you write the z as r e to the power i θ and this z as r cos θ + i r sin θ this is r cos θ + i r sin θ why I did this r e to the power i θ because there is outside there is ln. So, ln will take care of this. So, ϕ + i ψ here it would be. So, you have if I pull together all the real components and leave out the imaginary component from u z I have u r cos θ coming in u r cos θ that is the real part and m by 2 π ln z here m by 2 π ln r would be the real part and here the imaginary part would be m by 2 π ln of e to the power i θ ln of e to the power i θ is simply i θ.

So, it would be m by 2 π i θ that is the imaginary part and here I have i r u sin θ that is the imaginary part. So, now, I club all the real part within a bracket and imaginary part within a bracket. So, we see that the ϕ the real part would be u r cos θ that is the real part here I can see that u r cos θ and m by 2 π ln r that is the real part coming in here from here. So, ϕ is this and ψ is u r sin θ that is the imaginary part coming from here u r sin θ and ln of r e to the power i θ. So, from there that from there we have m by 2 π m by 2 π it should not be ln θ it should be θ it is wrong it is m by 2 π θ because m by 2 π i m by 2 π θ or m by 2 π i θ that is the imaginary part.

So, it would be it would not it is not ln θ it is m by 2 π θ. So, this is the ψ. So, if this is the stream function u r sin θ + m by 2 π θ. So, u r u r sin θ + m by 2 π θ. Now, I put what all θ is equal to π for the stagnation point and r is equal to m by 2 π u we have already found out this distance is m by 2 π u and this θ is equal to 180 degree θ is equal to π.

So, we put those values and the streamline that passes through the stagnation point if we if we write these u m by 2 π u sin π + m by 2 π into π. So, this gives me m by 2. So, what that means, is this ψ for body streamline is equal to m by 2 and any point. So, what that means, is any point on this body streamline will follow ψ is equal to m by 2. So, now, if we try to find out some arbitrary point here let us say some arbitrary point here.

So, that arbitrary point on the body streamline. So, it is not any arbitrary r θ, but it is I write this as this that is the body streamline that is the streamline that passes through stagnation point. So, that has r s and θs. So, that r s and θs there would be a relation and that relation is given here. What are we trying to say here that this any point on body streamline that will follow ψ is equal to m by 2. So, what that means, is we can go to the equation for streamline.

The equation for streamline is U r sin θ + m by 2 π θ. So, U r U r s sin θs + U r sin θ that is what we said U r sin θ and m by 2 π θ. So, m by 2 π θs that is equal to so, this is the value of ψ and that ψ has to be equal to m by 2. So, any point you pick up say let us say I pick up this point. So, this point will have this as the r s and now this as the θs.

So, any point you pick up on this body streamline that has to follow this equation everywhere the stream function value has to be equal to m by 2. So, if you simplify this what would be the r s in this case r s would be equal to m by 2 - m by 2 π θs divided by U sin θs. So, you can see here if you simplify here you get m by 2 π U sin θs and the numerator in that case would be π - θs. So, that is exactly what is this r s here.

So, this is the r s. So, if this is what for any θs this is the value of r s. So, we may like to know first of all what would be this height at on above the origin this is the place where the source is placed. So, in the y axis if I follow y axis from the point where the source is placed how far would be the fallout of the source. So, if I want to know this distance. So, this distance I can get by simply with the working with this r s θs and by imposing θs to be π by 2.

And similarly if someone wants to know at infinity that means if we continue these streamlines at infinity what would be the stretch how far this would be stretched. So, to do that what we do is we then in that case θs has to be I mean you can see θs is decreasing and at infinity θs would be equal to 0. So, I have to find out for value of θs equal to π by 2 and θs equal to 0. In these 2 cases what are the corresponding value of r s because that will give me what is this height and what is this r s and if I know this r s I can find out what would be we can call this y s because r s can be at every r s I can have corresponding y s because r θ system and x y system I can toggle between them. So, θs equal to π by 2 if you put so r s is equal to this and in fact if we may not work with r s any further we can directly go to y s.

$$r_s = \frac{m}{2\pi U}\frac{\pi - \theta_s}{\sin\theta_s}$$

So, y s is equal

$$y_s = \frac{m}{2\pi U}(\pi - \theta_s)$$

So, in that case we can say that this value is this height is this height is m by m this height is $\frac{m}{4U}$ and this height is $\frac{m}{2U}$.

What is m? The strength of the source meter cube per second per meter perpendicular to the screen. What is U? U is the uniform flow velocity again. So, this is U is in meter per second. You have here in this context. So, you can find out what is the shape of the envelope, or what is the size of the envelope where the source where there will be a fallout.

And at any point you want to know what is the velocity. So, you can immediately find out by you have the f expression you already have the expression for v r, v θ, u r ,u θ, and you can convert them to u and v. So, at any point, you can find out what is the velocity using this exercise. On top of that, one can find out the pressure at the surface, particularly if one is interested in using this concept as an airfoil. And this is just one simple case of superposition.

This is rank in half body, you can produce rank in full body. You can in fact flow past a cylinder. When it comes to there is a cylinder and there is a flow going in. So, flow past a cylinder when one wants to simulate that is done by superposing a uniform flow and a doublet. You have already seen the doublet.

You have already seen the doublet how it is done. And then, if you superpose a uniform flow with a doublet, then you can simulate the flow past a cylinder. So, similarly there are many such examples of superposition, but this is okay. Here, you have to apply some amount of intuition, but here, it is a straightforward source and uniform flow. And then it could be simply vortex and uniform flow.

It could be simply two vortices. You may have in fact I can think of say let us say I have a vortex here. I have a vortex here and I have a wall here. So, I have a wall here. I can expect that it is, so what I will do is I will simulate I will not have the I will not consider the wall; instead, I will consider another imaginary vortex exactly if it is x distance away. If the center of the vortex is x distance away here also I go x distance and put another vortex imaginary vortex and then that will simulate the effect of the wall.

So, these are quite common. If you have a line source and another point source you consider another imaginary source and simulate the effect of the wall. So, these are quite common in this context of superposition and complex flow. I believe this is all because I have other topics to talk about. So, this is all I have as far as this superposition is concerned. I will continue working on other topics and you will see that how these concepts will come into play there as well.

That is all as far as this lecture module is concerned. Thank you for your attention.