Momentum Transfer in Fluids
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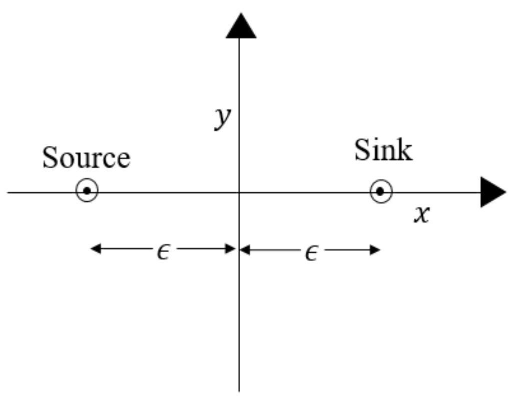
Lecture-34

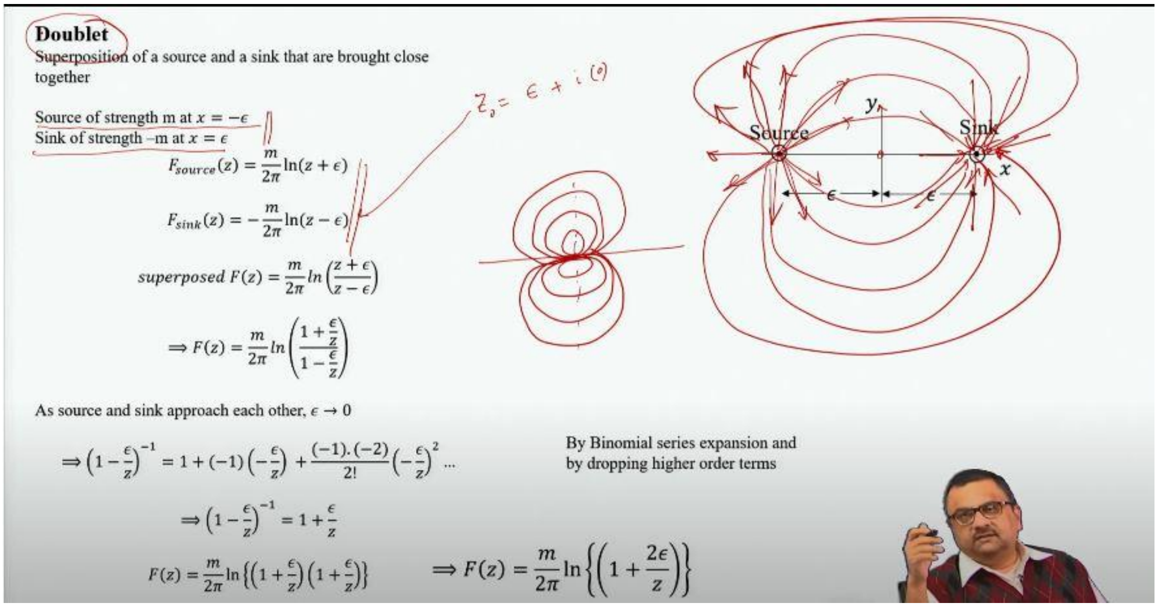
I welcome you to this lecture on Momentum Transfer in Fluids. We have been discussing complex potential and how to use complex potential to superpose various elemental flows and extract information out of it. So, we have already talked about uniform flow in x y direction at an angle and we have discussed about the source and sink and vortex. So, now, you have let us say I superpose source and sink and see where we get where we reach source and sink. So, superpose source and sink now what would be what would be a source we said it is z - z 0 right.



So, source is located here at x equal to source of strength m at x equal to - ϵ, source of strength m at x equal to - ϵ and sink of strength - m because strength would be again negative strength - m at x equal to ϵ . So, we are trying to superpose this two so that means, this is the y axis, this is the x axis and then we have located a sink here which is ϵ distance away from the origin this is the origin and another source here which is again ϵ distance away and the strength magnitude of the strength is same only since one is a source and the other is a sink. So, there is a - sign with the sink and we already said that it has to be z - z 0. So, when the source and sink they are not located at origin.

So, then there has to be a shift of coordinate. So, here it is z 0 in this case is ϵ is the z 0 what z 0 would be equal to ϵ + i 0 because why it is all lying on x axis itself. So, y component would be 0. So, z 0 in one case it is - ϵ another case z 0 is + ϵ. So, for the source it is z - z 0 and z 0 is again - ϵ.

So, - of - ϵ it becomes + ϵ and here it is $F_{sink}(z) = -\frac{m}{2\pi}\ln(z-\epsilon)$ because z 0 in this case is simply ϵ . So, this goes here. So, now I have these two sets. Now, we could have said that this just superpose these two. In fact, you can that exercise you can undertake right away if we try to superpose a source and a sink.



So, what type of streamlines would you expect? I mean let us try to draw this source will have these streamlines. So, these are the streamlines and the sink will have going towards this right. So, what type of streamlines will you see in this case? You will be seeing this to this, this to this, this to this like this right. So, you would expect and when the source and sink they are superposed like this right. So, you will have source and sink and if you have these.

So, you would expect that this type of structure will come in. Now, you can you yourself can work on that, but here we put another additional twist to it. The twist is that ϵ is tending to 0 that means, source and sink they are approaching origin, but they are not it is tending to 0 ϵ is not equal to 0. If ϵ is equal to 0, then source and sink both will fall on the origin itself and source is of strength m, sink is of - m they will cancel each other. So, everything is 0 no flow nothing, but it is not exactly that way we are saying ϵ tending to 0 that means, source and sink gradually you are you are reducing this distance you are gradually reducing this distance of ϵ .

So, what type of streamlines would you see? And let me tell you what type of streamlines you are going to see and then you will see that whether we reach there or not the streamlines that you are going to see is something like this. If this is the origin and this is what the origin is this is the x and y axis. So, you will see the net effect would be this type of flow if this type of streamlines you will end up with and this has a name to it this is known by the name it goes by the name doublet and doublet has its own doublet can be also considered as an elemental flow. Though doublet arises from superposition of source and sink with the with both source and sink approaching to each other, but then it is doublet is also I mean a lot of people treat doublet as an elemental flow and superpose it with other flows possible. So, this is a type of streamlines you are going to end up seeing.

How we end up there? is given by this and the sink is given by this. When you superpose then the superposed F(z) is equal to m by 2 π ln of a + ln of b. So, that would be equal to ln of ab and here there is a - sign. So, it went to the denominator. So, it becomes ln a by b ln a and - ln b it becomes ln a by b.

$$F_{source}(z) = \frac{m}{2\pi}\ln(z+\epsilon)$$

$$F_{sink}(z) = -\frac{m}{2\pi}\ln(z-\epsilon)$$

$$superposed\ F(z) = \frac{m}{2\pi} ln\left(\frac{z+\epsilon}{z-\epsilon}\right)$$

$$\Rightarrow F(z) = \frac{m}{2\pi} ln\left(\frac{1+\frac{\epsilon}{z}}{1-\frac{\epsilon}{z}}\right)$$

So, it becomes z + ϵ divided by z - ϵ . So, we can further simplify m by 2 π ln and then this is 1 + ϵ by z. I mean if I divide z divide numerator and denominator by z, then this numerator becomes 1 + ϵ by z and denominator becomes 1 - ϵ by z. So, as source and sink approach each other that means, as ϵ tends to 0, we can see up to this point we have not assumed ϵ tending to 0, but beyond this what we do is 1 - ϵ by z. So, this we can write as m by 2 π into ln of 1 + ϵ by z into 1 - ϵ by z inverse.

And then if we focus on the inverse part 1 - ϵ by z to the power - 1, if we focus on this part and if we take a binomial series expansion and if we drop higher order terms 1 - ϵ by z to the power - 1 gives me 1 + ϵ by z. So, then it is m by 2 π ln. So, this 1 - ϵ by z to the power - 1, these can be replaced by 1 + ϵ by z when ϵ tending ϵ tends to 0. So, then this is 1 + ϵ by z. So, ln of 1 + ϵ by z into 1 + ϵ by z.

So, that becomes equal to again since we have already assumed ϵ tending to 0 and we ignored higher order terms. So, similarly here ln of 1 + ϵ by z whole square. So, that will have what? 1 + ϵ by z whole square will have 1 + 2 ϵ by z + ϵ by z whole square. So, that is what we are going to see. So, out of that since we have ignored ϵ square term in previous Taylor series expansion.

As source and sink approach each other, $\epsilon \to 0$

$$\Rightarrow \left(1-\frac{\epsilon}{z}\right)^{-1} = 1+(-1)\left(-\frac{\epsilon}{z}\right) + \frac{-1.-2}{2!}\left(-\frac{\epsilon}{z}\right)^{2} \ldots$$

$$\Rightarrow \left(1-\frac{\epsilon}{z}\right)^{-1} = 1+\frac{\epsilon}{z}$$

$$F(z) = \frac{m}{2\pi}\ln\left\{\left(1+\frac{\epsilon}{z}\right)\left(1+\frac{\epsilon}{z}\right)\right\}$$

$$\Rightarrow F(z) = \frac{m}{2\pi}\ln\left\{\left(1+\frac{2\epsilon}{z}\right)\right\}$$

So, here also we ignore this and write 1 + 2 ϵ by z. So, if z can be written as m by 2 π ln 1 + 2 ϵ by z. So, that becomes equal to F(z). So, now if we work with so, we have ln of m by 2 π ln of some quantity ln of x when the x is small ln of x can be written as this expression and then ln of 1 + x since it is 1 + 2 ϵ by z. So, ln of 1 + x can be written as $x-\frac{x^2}{2}+\frac{x^3}{3}$ like this.

So, this in this series. So, ln of 1 + 2 ϵ by z can be written as m by 2 π and if I work only with the x and ignore the higher order terms it would be m by 2 π into 2 ϵ by z. So, F(z) becomes equal to m this 2 and this 2 will cancel out. So, m ϵ by π z. So, this is equal to F(z).

$$\ln x = (x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3$$

$$ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}$$

$$\Rightarrow F(z) = \frac{m}{2\pi}\left(2\frac{\epsilon}{z}+\cdots\right)$$

$$\Rightarrow F(z) = \frac{m\epsilon}{\pi z} = \frac{\mu}{z}$$

So, F(z) becomes equal to if m ϵ by π if μ is defined as m ϵ by π. So, then this can be defined F(z) can be defined as μ by z. In fact, that is what that is how it is customarily written just like we had for a source or sink we have $F(z)$ is equal to cln z and c was m by 2 π. So, in this case for a doublet F(z) is written as μ by z where μ is simply m ϵ by π where m is the strength of source or - m is the strength of the sink which are approaching to each other and the ϵ is the distance between the two to distance from the origin distance of the source from the origin and π m ϵ by π. So, this is one complex potential μ by z which defines doublet.

Now, how do you know I mean what type of stream what type of flow what type of streamlines this will result to. We know that let us say now onwards because m is a constant ϵ and π let us say if we work with μ only. So, we see that the F(z) is given by

$$F(z) = \frac{\mu}{r} e^{-i\theta} = \frac{\mu}{r}(\cos\theta - i\sin\theta)$$

Now, for a x y system I mean if you go back to our x y coordinate system cos θ - i sin θ if this is x and this is y. So, cos θ - i sin θ cos θ is what in our x y coordinate system cos θ is basically cos θ this angle is θ this is x, this is y this is coordinate I can write it as r θ system or I can write it as x y system. So, cos θ is essentially this divided by this that is what the definition of cos θ is. So, this is x and this is square root of x square + y square.

$$F(z) = \frac{\mu}{r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + i\left[-\frac{\mu}{r}\frac{y}{\sqrt{x^2+y^2}}\right]$$

We have on the other hand f z is equal to ϕ + i ψ.

So, if I leave this out and write - i and sin θ is y divided by square root of x square + y square. So, now, I can see here that this i is this r itself is square root of x square + y square this r itself is square root of x square + y square. So, what if I write try to write f z as ϕ + i ψ then ϕ would be equal to

$$\phi = \frac{\mu}{\sqrt{x^2+y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$\phi = \frac{\mu x}{x^2+y^2}$$

ψ would be equal to μ into y divided by x square + y square, but I see a - sign coming in here.

$$\psi = \frac{\mu}{\sqrt{x^2+y^2}}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\psi = \frac{\mu y}{x^2+y^2}$$

So, now, if I try to do this if this is ϕ and this is ψ and then we try to work with the ψ here. So, then what do I get? If I write this here this comes to if I simplify this further, it would be x square into ψ + y square into ψ that is equal to that is equal to - μ y. That is that is if we try to simplify this or in other words, I can take ψ out of this. So, this becomes equal to x square remains as it is ψ is going there to that side + y square that is equal to - μ y divided by ψ. So, if I try to, my point is that you will end up with this expression from these because if you add y square, if you add here +, let us say I bring this μ y by ψ towards this side. So, then what do I get? I get x square x square I am writing it here itself x square + y square + μ y by ψ that is equal to 0 or we can add to both new both left hand side and right hand side we can add let us say μ by 2 ψ whole square to the left hand side and to the right hand side also μ by 2 ψ whole square.

$$x^2 + \left(y + \frac{\mu}{2\psi}\right)^2 = \left(\frac{\mu}{2\psi}\right)^2$$

So, what do I get on the left hand side? Left hand side I will be getting x square will remain as it is + y square + μ by 2 ψ whole square + 2 into y into $\frac{\mu}{2\psi}$ then this 2 and 2 will cancel out you will get μ y by ψ. So, you will get y + μ by 2 ψ whole square. So, the left hand side out of this part you will end up with x square + y by μ by ψ whole square and right hand side you will be left with $\frac{\mu}{2\psi}$ whole square. So, that is the expression you will end up here. So, forget about this part and this part if you simply follow this with ϕ + i ψ and then you will write ϕ as this and ψ as this and then if I focus only on ψ and then I write correspondingly if I write this expression and then I go to this expression and this I add one additional term $\frac{\mu}{2\psi}$ whole square I take this to the left hand side and I add additional term $\frac{\mu}{2\psi}$ whole square to both left hand side and right hand side the left hand side becomes x square + this whole square and the right hand side becomes $\frac{\mu}{2\psi}$ whole square.

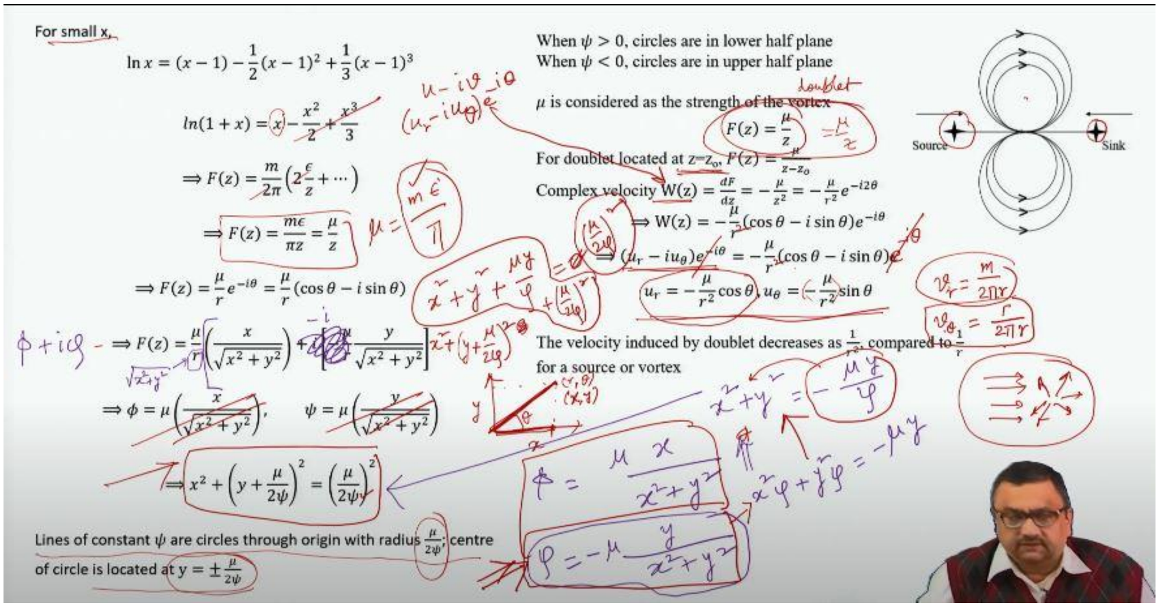
Where are we heading to? What we are trying to say here is that the lines of constant ψ are the streamlines. So, lines of constant ψ here are in this case lines of constant ψ if ψ is becoming constant what kind of lines are they talking about it is these equation if you follow coordinate geometry this is the equation for circles, but circles these are for a constant ψ I will get a circle which is having the circles they have one unique trait you can see the circles will be passing all circles will pass through 0 0 that is one thing all circles will pass through 0 0, but x equal to 0 y equal to 0 it is satisfied. So, all circles will pass through 0 0 circles will have radius of $\frac{\mu}{2\psi}$ right x square + y by s b whole square is equal to b square. So, that means, these are this is the radius $\frac{\mu}{2\psi}$ is the radius of the circle and the origin of the or the center of the circle the center of the circle is located at x equal to 0 and y is equal to - $\frac{\mu}{2\psi}$ that is what this states. So, that is why I am saying that these would be circles all circles are passing through the origin the all circles are passing through 0 0 and center of the circle gets shifted as I take different values of ψ the center of the values get shifted.

So, these are all unique lines these are all lines along which the ψ value is constant. So, I increase or I change the ψ the radius changes as well as the center of the circle shifts, but in both cases the shift of the or the distance from the origin to the center of the circle is same as the radius of the circle. So, all circles are passing through 0 0 and they are getting shifted like this. So, we had the source. We had the sink, and they were falling on each other. That is, f's are tending to 0, but ψ is small primarily. When ψ is small, we end up with this type of flow, which is referred as doublet. So, lines of constant ψ are circled through the origin with radius $\pm\frac{\mu}{2\psi}$ centers of the circle is located at y equal to $\pm\frac{\mu}{2\psi}$ it could be the positive side of the origin, or it could be on the y axis, or it could be negative side in the y axis.

When ψ is greater than 0 circles are in the lower half plane when the ψ is greater than 0 ψ is greater than 0 means ψ is positive when ψ is positive then y + this. So, then this is the shift of center of the circle is negative. So, that means circles are in the lower half plane, and when ψ is less than 0, the shift would be on the positive side. So, that is why the circles will be on the upper half plane. μ is considered as the strength of the doublet.

For doublet located at z equal to z 0 that means, if there is again doublet is here and another source is there and you are trying to comπle or doublet is there vortex is in at some other place and you are trying to superpose then they are also the you have to do this z equal to z 0 f z is equal to μ by z - z 0. So, here in this case if someone wants to know the velocities because the complex velocity W(z) you need to find out what is the complex velocity W z and that is equal to dF dz. If you draw dF dz on this, that means, if you take a derivative of this with respect to z, it would be - μ by z square. I mean a derivative of 1 by z is - 1 by z square. So, it would be - μ by z square, and then again z is again what z is equal to $re^{i\theta}$, z is r cos θ + i r sin θ again Euler theorem. So, $re^{i\theta}$ whole square so that means, r square e to the power 2 i θ, 2 i θ goes to the numerator.

So, it becomes $e^{-i2\theta}$. So, W(z) the complex velocity in this case becomes μ by r square - μ by r square $e^{-i2\theta}$. So, now, if you do this because you have to find out some you have to break it into real and imaginary component and you know very well that this W(z) in turn is linked to u - i v that is what I have been doing all along in for earlier cases and u - i v we have already demonstrated that there is a vector identity which says it is u r - i u θ $e^{-i\theta}$ and we want to use this expression because here at least earlier we had this $e^{-i\theta}$ on the other side and we could cancel it, but here it is $e^{-i2\theta}$. So, at least 1 i θ we can cancel.



So, W by z here is μ by r square. So, what we did is anyway we can cancel 1 $e^{-i\theta}$. So, let us keep this $e^{-i\theta}$ here as it is and other $e^{-i\theta}$we can convert it back to cos θ - i sin θ. So, cos θ - i sin θ remains here - μ by r square cos θ - i sin θ $e^{-i\theta}$. Then, now we will link it when when we related to u r - i u θ $e^{-i\theta}$. So, this μ by r square cos θ - i sin θ $e^{-i\theta}$. For doublet located at z=z$_o$, $F(z) = \frac{\mu}{z - z_o}$

Complex velocity W(z) $= \frac{dF}{dz} = -\frac{\mu}{z^2} = -\frac{\mu}{r^2} e^{-i2\theta}$

$$\Longrightarrow \mathrm{W(z)} = -\frac{\mu}{r^2}(\cos\theta - i\sin\theta)e^{-i\theta}$$

$$\Longrightarrow (u_r - iu_\theta)e^{-i\theta} = -\frac{\mu}{r^2}(\cos\theta - i\sin\theta)e^{-i\theta}$$

$$u_r = -\frac{\mu}{r^2}\cos\theta\,,\; u_\theta = -\frac{\mu}{r^2}\sin\theta$$

So, this is what is u r and u θ. You can see that the velocity induced by doublet decreases as 1 by r square. In case of source or vortex it was in case of source the strength was in case of source the velocity was we said velocity is equal to c by r and c was m by 2 π. So, it was m by 2 π r in case of source. In case of vortex v θ was Γ by 2 π what was the v θ in there it Γ by 2 π ln z and then we have v θ was v θ is Γ by 2 π r.

So, that is what is v θ. So, that means, v r is proportional to 1 by r v θ is proportional to 1 by r. So, it was v θ they are all decreasing proportional to 1 by r means as r increases v r decreases source the radial velocity decreases with r vortex the θ component v r is 0 for vortex, but v θ is decreasing as I go away from the center of the vortex. Here, in this case, it is decreasing, but as a function of 1 by r square, that means the velocity induced by a doublet decreases μch faster. It decreases μch faster than the decrease in velocity in case of source or vortex. So, that is one aspect of it. Now, this doublet would be the doublet is considered as just as an as another elemental flow whose complex potential is given by μ by z complex potential is given by μ by z.

So, that is how it is, but it has these traits and the streamlines for a doublet appears like this. I will work on another example of truly an example of superposition where it is not just an elemental flow rather superposition of elemental flow and I will try to see what all information we can extract out of it. So, in the next lecture I will work on retro superposition and show how streamlines will look like. In fact, my plan is to go for a superposition of uniform flow and you have a source and if you do a superposition then how do you define the streamlines in that case and how do you extract the information of velocities etcetera. So, that is something that I will do in the next class and continue from there, but more or less, the essential components or all the elemental flows for this complex potential have already been talked about. Only now do I have to give some true examples of superposition.

That is all I have for this lecture module. Thank you very much for your attention.