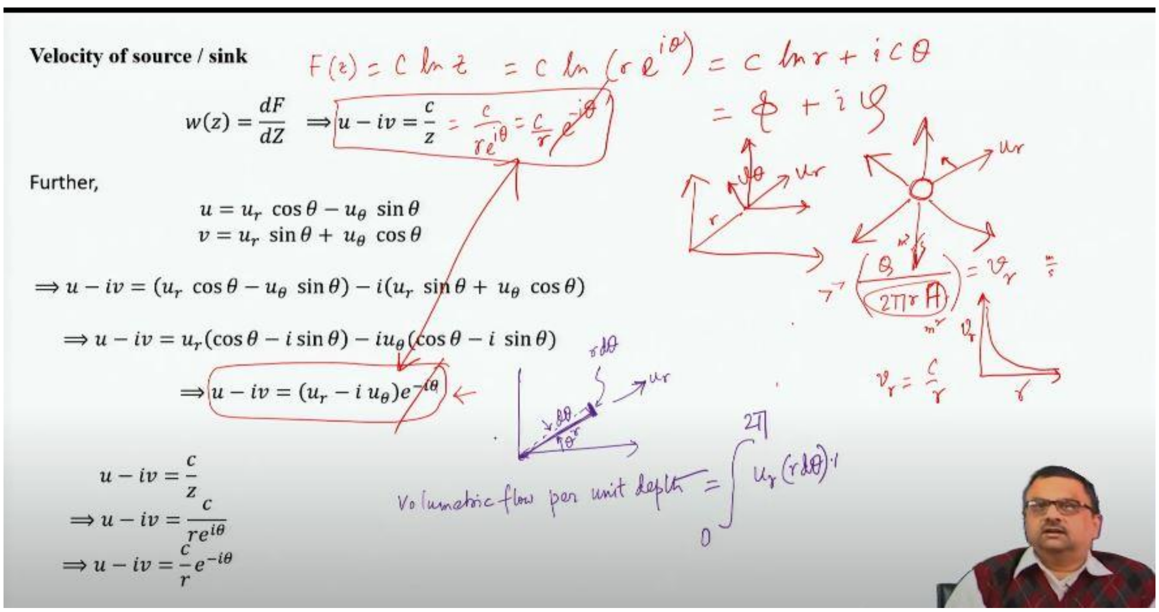
Momentum Transfer in Fluids
Prof. Somenath Ganguly
Department of Chemical Engineering
IIT Kharagpur
Week-07
Lecture-33

I welcome you to this lecture module on Momentum Transfer in Fluids. We have been discussing the complex potential, and in particular, we were talking about various elemental flows. We had discussed about this source or sink and there we have seen that if we try to find out the we know that this source or sink is given by F(z) equal to c ln z because we have seen that this ln z is this will take me to c ln Re to the power i θ and then this will break into c ln r plus c ln e to the power i θ. So, that would I write as i c θ and then I can see that this is equal to ϕ plus i ψ. So, we see that the ψ is along the streamline ψ is constant and for this F(z) ψ is equal to c θ.

That means, in this case along the streamline c θ is constant. So, the lines for which θ is constant that represents streamlines here. So, that is why it is radial flow either outward or inward depending on the sign of c. That is what we already knew, and on top of that, we are trying to relate trying to find what is the physical meaning of c. So, in this context we have established some vector identity.



One is that z equal to $re^{i\theta}$ that is Euler theorem we are now using interchangeably. The other vector identity we have seen is this u-iv is equal to ur-i u θ e to the power-i θ where u if this is the Cartesian system. So, then if this is r and if this is ur and perpendicular to this is u θ then you have u and v. So, then u and v will be related to ur and u θ by this vector identity. So, now, with this understanding if we try to find out the meaning of c first of all this when we equate it is c by z $\frac{dF}{dZ}$ is equal to $\frac{c}{z}$ because c ln z when you take the derivative of it derivative of ln z is 1 by z.

So, u-iv is $\frac{c}{z}$ and z we are writing as $re^{i\theta}$. So, this is taking me to c by r e to the power-i θ and u that is equal to u-iv. So, u-iv is equal to c by r e to the power-i θ and u-iv is ur-i u θ e to the power-i θ. When we equate these two immediately we see that when we equate this with this I can immediately cancel e to the power-i θ and I see here that this is here I am left with c by r. So, what I am getting is ur-i u θ that is equal to c by r.

So, what does this mean? This means that here the right hand side is real and left hand side ur-i u θ. So, that means, u θ has to be equal to 0, I compare real with real left and right left side and right hand side real with real imaginary with imaginary. So, I see here that the u θ is equal to 0 and ur is equal to c by r which makes perfect sense because you can see here u θ is equal to 0 for a source or sink. Source or sink means flow radially let us see if I talk about source flow radially outward. So, here in this case I do u θ exists u if this is ur u θ would be perpendicular to this there is no flow perpendicular to a streamline streamlines are radially outward lines and that is one of the condition that no flow can exist perpendicular or cross the no flow can exist perpendicular to a streamline.

So, this is u θ is equal to 0 it makes perfect sense and c by r. Now, c by r you can quickly see the quickly think about what this c could be when you have a flow outward let us say here I am talking about flow unit depth perpendicular to the screen. What does unit depth perpendicular to the screen means if I try to draw a 3 dimensional view of this. So, this is the central part let us say and then from there flow is radially outward. So, over if this is the unit depth this represents the unit depth I am looking at the third from 3 dimensional view.

So, from everywhere the flow is emanating right everywhere flow is emanating it is like you have a small let us say you take a straw and then make holes pinch put lot of holes on that straw and then blow water through it you will find water would be oozing out from all sides as far as that straw is concerned let us say this is the straw. So, this from there everywhere the flow would be coming out in all directions. So, that is essentially a source with the depth equal to the length of that straw. So, here we are talking about a 2D flow and we are considering per unit depth perpendicular to the screen. Essentially it is something like this that straw you generally ignore the pressure drop across the straw flow takes place through these holes and the pressure drop for going out through these holes that that dominates, but that is a different thing.

Let us say if you have a flow like this would be a perfect case of as long as this depth is unit depth that is if it is in meter 1 meter. So, then that represents a source. So, what would be this perimeter if radius is r let us say of that straw I am talking about if the radius is r then if the flow that is taking place that is Q. Let us say I am talking about a far away place here this is the radius. So, and then I can have another radius some at some other r.

So, let us say or actually the flow is along a even over a much smaller cylinder inside. So, I am taking any cylindrical element if I take and then what would be the area through which the flow is taking place? This is 2πr this is the perimeter multiplied by this height if I put this height H I mean H is going to be unit depth, but let us say I put this put this value as H. So, total area over which the flow is taking place if I take that as the reference it would be the area would be 2πr H that is the area 2πr is a perimeter multiplied by the H that is the area of this cylinder. So, 2πr H over which this flow is taking place and what is the total flow rate that is Q and that Q remains same whether it is here whether it is far away the flow remains same. So, it is Q by 2πr H that gives me the value of V r the velocity flow rate in meter cube per second and this is meter square r is in meter H is in meter.

So, this is meter square meter cube per second divided by meter square that gives me V r in meter per second. So, that is essentially the velocity. So, you can see here that the v r is equal to if H is unit depth you take it or and Q is remaining fixed constant flow rate if the source is source has a constant flow rate by which the source is being fed. So, V r would be eventually c by r because r is in the denominator. So, the point is the velocity near the source the velocity would be highest the radial velocity would be highest as you go away from the source the radial velocity decreases and decreases very fast.

In fact, if you plot V r versus r you will find that the velocity decreases very fast because V r is equal to some constant divided by r. So, if we forget about these exercise just by common intuition we see here as the radial velocity would take C by r form and that is exactly what we see here. Now, what is this how still our point is my point our point is what these how we relate this C. So, here you see here that in fact, you can see here that what is C, C would be here Q by 2 π H, Q by 2 π H would be the C I mean if you look at, but I have a better way to handle this let us see how we can handle it. So, u-i v we have

$$u - iv = \frac{c}{z}$$

$$\Rightarrow u - iv = \frac{c}{re^{i\theta}}$$

$$\Rightarrow u - iv = \frac{c}{r}e^{-i\theta}$$

Now, volume flow per unit depth if I want to write volume flow per unit depth see it is something like this volume flow per unit depth means let us say we have this is my x y system and this is the angle θ and this is the u r that you have. So, this u r would be valid over a differential arc of this angle is dθ. So, this length of this arc would be in that case if this distance is r then this length of this arc would be r dθ and then if we assume this to be perpendicular to the screen unit depth. So, in this case it will not be H that we are talking about in unit depth. So, if we are trying to find out what is the volumetric flow per unit depth, if we try to find out what is volumetric flow per unit depth, then we will write u r is the velocity here, u r is the velocity here.

So, u r velocity multiplied by the area over which u r operates. So, that means, r dθ into unit depth r dθ is this arc length and unit depth perpendicular to the screen. So, 1. So, r dθ into 1 into u r. So, volumetric flow per unit depth would be in this case u r and into R dθ and into 1 unit depth and then that has to be integrated over all values of θ from 0 to 2 π then you get the complete volumetric flow per unit depth as far as the source is concerned.

$$\int_0^{2\pi} u_r\left(r\, d\theta\right) = 2\pi c$$

So, that is exactly what is being done u r into R dθ integration between 0 to 2 π and u r is we have already seen u r is c by R that is that is what we have noted. When we cancel this e to the power-i θ and e to the power i θ when we cancel this. So, then we left with u r-i u θ is equal to c by R and so, we said u r is equal to c by R. So, instead of u r we are putting c by R. So, c by R this R and this R will cancel out.

So, it is c goes outside integration dθ. So, 0 to 2 π this gives me 2 π c. So, then what is c then c is volumetric if volumetric flow per unit depth that means, unit would be meter cube per second per meter perpendicular to the screen. So, if that is what volumetric flow what is flow means flow rate I am talking about obviously, flow rate per unit depth. So, then in that case if that is given as m let us say.

So, we see that m is equal to we see that m is equal to 2 π c or c is equal to m by 2 π. So, instead of f z equal to c ln z that is what we have said that that gives me a source or sink instead of that we can we can replace the c by m by 2 π where m is the what we call the strength of the source and strength of the source is given by volumetric flow rate per unit depth perpendicular to the screen or perpendicular to the to the you are working in a 2D platform perpendicular to the platform on which you are working. So, this now I have linked now I have I could get rid of c and now I have brought in the meaningful term which is m. Obviously, if you have instead of source if you have a sink then the m would be m will take a negative value here m is a positive one there the m will be m will take a negative value. So, this is this is how we will write this f z and on top of that as I said that all your source and what x and everything will not be at the same location they would be at different locations.

$$\Rightarrow F(z) = c\ln(z)$$

$$\Rightarrow F(z) = \frac{m}{2\pi}\ln(z)$$

When the source is located at position $z=z_o$

$$F(z) = \frac{m}{2\pi}\ln(z - z_o)$$

Here we have assumed that the source or sink is located at the origin 0 0, but if it is not if it is shifted then in that case you will write this as m by 2 π ln z-z 0 where z 0 is the location of that source or sink. So, if source or sink is not located at a particular at the origin at 0 0 then there has to be a shift of coordinate and for that reason you have a z-z 0. So, z 0 will be again x 0 plus i y 0 x 0 is the x coordinate of that location where the source is and y 0 is the y coordinate of the location where the source is x and y you are calculating from the origin. So, this is another point which you must note and this would be applicable for all other elemental flows as well. So, now we have a meaningful now we have some C we have some meaningful parameter instead of C through which we can work we can we can now we can we can work on and because volumetric flow per unit depth is something which is meaningful just like you have velocity u or velocity v.

$$F(z) = -i\,c\ln(z)\,,\ (\because c \in \mathbb{R})$$

$$\Rightarrow F(z) = -i\,c\ln(re^{i\theta})$$

$$\Rightarrow F(z) = c\theta - ic\ln(r)$$

$$\Rightarrow F(z) = \phi + i\psi$$

So, this is a very similar parameter. Now, the vortex we said that f z is equal to -i c ln z. So, we mentioned there that f z is equal to -i c ln z that is the vortex that is the definition of vortex and where C is a real constant. So, now you have f z is -i c ln z obviously, for both source and sink and the vortex there is no point in working with the Cartesian system. So, immediately its z is replaced by r e to the power i θ through that identity we have been talking about and through use of wireless theorem.

So, then you ln of product AB is ln A plus ln B. So, you break it up ln r is showing up here as-i c ln r and e to the power ln of e to the power i θ would be only i θ and that i will multiply with this i to become-i square and that becomes plus. So, f z is C θ-i c ln r. So, now, this is f z in turn is equal to ϕ plus i ψ. So, this is ϕ and this corresponds to ψ that is so, line of constant ψ is the line along which r is constant.

So, we have already discussed the streamlines are circles and equipotential lines are the radial lines in this case. So, this indicates a vortex. Obviously, we said that these are irrotational vortex that means, at the very core of the vortex this equation may not apply because there it would be a viscous core would be present. Now, in this case now if we try to find now here what is the meaning of this c, I mean there I have in case of in case of source or sink we have we could make some meaning out of that instead of c we started using m by 2 π where m is the volumetric flow rate per unit depth perpendicular to the screen or in case you have a third dimension volumetric flow rate per unit depth in the third dimension that is what it is. Now, here in this case if we try to find some meaning of C i g here that the W is equal to dF dz and that is equal to U-i V this is our standard vector identity we have been talking about so far and then dF dz when I take when I take dF dz of this it is obviously,-i C and d dz of ln z is 1 by z so,-i C by z.

So, that is what I have written here-i C by z. So, here in this case now this I have already mentioned that this is a vector identity d U-i V can be replaced by U r-i θ e to the power-i θ you have we had this discussion earlier where we said that U-i V is once we wrote first of all U and V these are U r cos θ-U θ sin θ and V is this. So, we have already worked on this we and this is the vector identity we have we have already mentioned earlier. So, the same exercise we can we can do here instead of repeating those steps I am just skipping those steps and I am writing that directly u-i v is equal to U r-i U θ e to the power-i θ. So, u r is if this if this is my if this is the point this is the U r and this is the U θ.

So, then this is the this is the U and this is the V and then this angle is θ. So, we wrote U r cos θ U θ sin θ etcetera and then did some manipulation and we arrive at this expression this is a vector identity irrespective of what elemental flow we consider. So, if we if we take this as U-i V and on the right hand side you remember what we did in case of source or sink this Z was replaced by r e to the power i θ and that is equal to-i c by r e to the power-i θ went in the went to the numerator. The objective is that this e to the power-i θ and this e to the power-i θ we can cancel and then we can now we have this is an imaginary number here I have U r and i U θ. So, now, I can link real with real imaginary to imaginary exactly the way what we have done in case of source and sink only in case of source or sink it was simply c by z and here in case of vortex it is-i c by Z because in case of source or sink f Z was c l n z and here in case of vortex we said f Z is equal to-i c l n z.

$$W(z) = \frac{dF}{dz} = u - iv = -i\frac{c}{z}$$

Or $(u_r - iu_\theta)e^{-i\theta} = -i\frac{c}{z}$

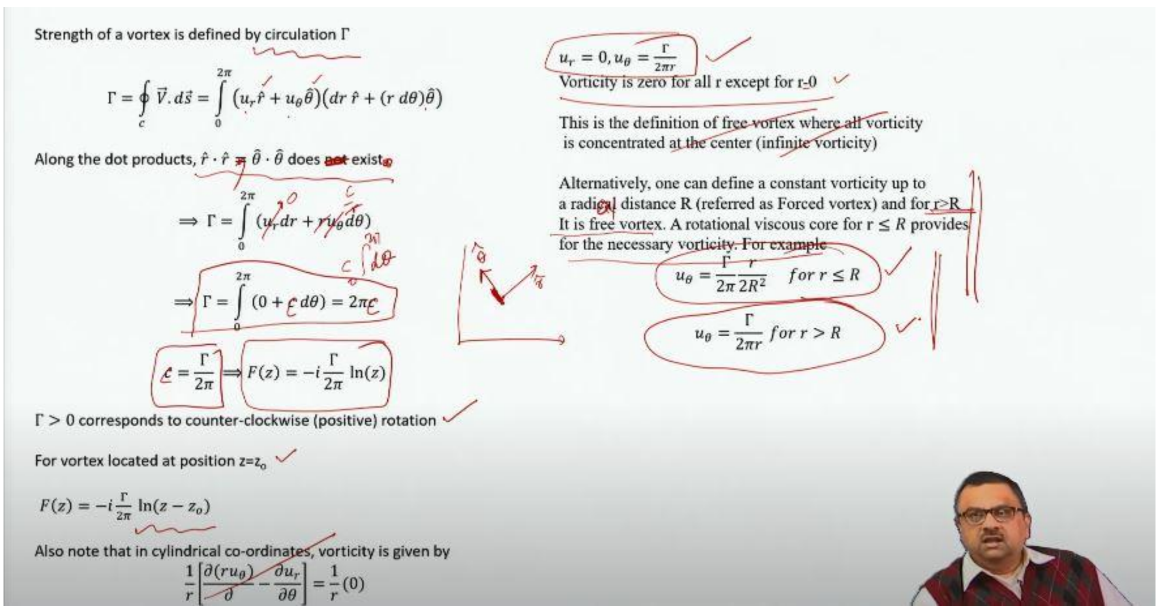
$$(u_r - iu_\theta)e^{-i\theta} = -i\frac{c}{re^{i\theta}}$$

$$(u_r - iu_\theta)e^{-i\theta} = -i\frac{c}{r}e^{-i\theta}$$

$$\Longrightarrow u_r = 0, \quad u_\theta = \frac{c}{r}$$

C > 0 implies counter-clockwise (+ve) rotation

So, then if it is U r is equal to 0 and U θ is equal to c by r what does c mean in that case? How we can link? So, in fact, this is what it says and last time we said c is greater than 0 means it is a source c is less than 0 means it is a sink.



Here in this case also c greater than 0 means it is the rotation is counterclockwise that is counterclockwise is treated as positive and c is negative means rotation is clockwise. So, the vortex is rotating in a clockwise manner when c is negative. But how to link this c? So, what we do here is we try to find out circulation Γ because circulation Γ is something which is which we are which I we said that this circulation is something which is existent which is a better description than vorticity. In fact, now it turns out that why circulation is important because we could not for a for a flow in circular streamline which we are calling vortex in this case in as vortex as an elemental flow we could not have we could have the outright said vorticity is 0, but that is not true it must have some strength like source has a strength which is m by 2 π where m is volumetric flow rate per unit depth. So, similarly here also vortex has a strength.

Strength of a vortex is defined by circulation Γ

$$\Gamma = \oint_c \vec{V}.d\vec{s} = \int_0^{2\pi} (u_r\hat{r} + u_\theta\hat{\theta})(dr\,\hat{r} + (r\,d\theta)\hat{\theta})$$

Along the dot products, $\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = 1$

$$\Rightarrow \Gamma = \int_0^{2\pi} (u_r dr + ru_\theta d\theta)$$

$$\Rightarrow \Gamma = \int_0^{2\pi} (0 + c\,d\theta) = 2\pi C$$

$$C = \frac{\Gamma}{2\pi} \Rightarrow F(z) = -i\frac{\Gamma}{2\pi}\ln(z)$$

$\Gamma > 0$ corresponds to counter-clockwise (positive) rotation

For vortex located at position $z=z_o$

$$F(z) = -i\frac{\Gamma}{2\pi}\ln(z - z_o)$$

So, circulation is now coming into play we can in terms of circulation we can explain what is c would be related to circulation. So, in this case for this vortex the circulation is simply integral v dot ds that was the definition of circulation we had we had in earlier lectures we have already discussed this over the entire circle and of course, it is anticlockwise. So, now, v is written as $(u_r\hat{r} + u_\theta\hat{\theta})$ that means, you have you have you have both you have these are the coordinate system and then you have this is the u r and this is the this is the r hat component and this is the θ hat component. So, you will write v as $(u_r\hat{r} + u_\theta\hat{\theta})$. So, instead of i and j we are using unit vector r and r hat and θ hat and it along the r hat the coordinate is u r and along the θ hat it is u θ and for ds is that length.

So, that would be having along the r hat it is $dr\,\hat{r} + (r\,d\theta)\hat{\theta}$ I mean if you take if you take a if you look at the ds, ds would take a shape of $dr\,\hat{r}$, dr is the differential length in the direction of r and r dθ is the arc length in the direction of θ. So, you will write this as v and this as ds and take their dot product and do this integration for between 0 to 2 π the entire full circle that will give you the Γ. Now, if you are looking at the dot product then these does not exist and these are and these are and you are ending up with no you this is not true you what you are seeing here is u r dr plus r u θ dθ. So, this r hat dot r hat andθ dot θ hat exists.

$$\oint_c \vec{V}.d\vec{s} = \int_0^{2\pi} (u_r\hat{r} + u_\theta\hat{\theta})(dr\,\hat{r} + (r\,d\theta)\hat{\theta})$$

So, if you do this and then in this case you if you write what is u r, u r in this case is 0 and u θ is equal to c by r this we already have seen in case of vortex.

So, you are writing u r is 0 and u θ is c by r. So, since we are not anyway considering r equal to 0 because that is r equal to 0 this definition does not exist. So, I can cancel r and r. So, this becomes c dθ. So, Γ becomes equal to integration 0 to 2 π 0 plus c dθ.

So, essentially this is equal to c integration 0 to 2 π dθ. So, this is equal to 2 π c. So, you can see we have linked this Γ with c this is capital C mind it. So, c is equal to Γ by 2 π.

So, this is the new definition. Now, I have linked c with a meaningful parameter. So, you have in case of source or sink it was m by 2 π where m is volumetric flow rate per unit depth and here in this case the c is Γ by 2 π where Γ is the circulation. So, now, we have a meaningful circulation is something meaningful which we can relate to in a physical quantity. So, instead of f z equal to-i c ln z now you will write $F(z) = -i\frac{\Gamma}{2\pi}\ln(z)$. So, now, you can when Γ is greater than 0 that corresponds to counter clockwise rotation that is understood for vortex located at position z equal to z 0.

I mean that means, when the vortex is not located at the origin then there will be a shift of coordinate as we mentioned in case of source and sink. So, that is what we have here there would be ln of z-z 0 that has to be accounted. Cylindrical coordinates there you will have a similarly this is not we are not I mean what we can do is here we have u bar is equal to 0 and u θ equal to Γ by 2πr that is all very fine and what is it is 0 fine that is understood here. So, here we are noting that the definition of free vortex is what? The definition of where all vorticity is concentrated at the center basically it is vorticity, vorticity is 0 for all r except for r equal to 0 that is true because if r is equal to 0 then you could not do this manipulation here r is equal to 0 means you cannot cancel r by r in the denominator r 0 means it becomes infinite. Generally, it will leave this out generally this part a constant vorticity up to a radial distance up to a radial distance r is referred as force vortex and when r is greater than r this I have mentioned before there is a threshold r then it is a free vortex.

So, a rotational viscous core for $r \leq R$ provides for the necessary vorticity. So, you can have accordingly for $r < R$ you have one relation in terms of Γ again circulation and when you have r > R you have another equation for u θ in terms of Γ once again. So, this is for a viscous core and this is for the outside the viscous core the vortex that is applicable. So, this is more or less so bottom line is we could get some meaningful physical significance of C, $C = \frac{\Gamma}{2\pi}$ where Γ is circulation and circulation gives the strength of a vortex. So, C is linked to the strength of the vortex that is all I have as far as this lecture module is concerned. Thank you for your attention.