Momentum Transfer in Fluids
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I welcome you to this course of Momentum Transfer in Fluids. We have been discussing a concept called complex potential which is a combination of potential function and stream function and I will continue that discussion. What we do here, the objective of this exercise is to utilize this concept of superposition. As I mentioned in the end of the last class that if we have various elemental flows, let us say some elemental flow which is uniform flow here in x direction. At the same time I have a source term here.

So, some flow is taking place like this, and at the same time here, there is a vortex continuing. So, if we have a situation similar to this, then how will you find it, how the streamlines will look like. So, streamline for the uniform flow would be this, streamlines for vortex would be circles and streamline for source would be lines radially coming outward. So, when you have the combined effect, how do you obtain the streamlines, how do you obtain the velocity at various points.

So, that is the exercise where we are trying to get into. So, in this respect, I need to define something called a complex potential which I have mentioned very briefly before that $F(z) = \phi(x, y) + i\,\psi(x, y)$, because we have seen that ϕ and ψ the stream function, streamlines and potential lines they are orthogonal to each other. So, they have that property we can utilize this by writing it in terms of a complex potential. Now, we know that u is equal to $\Delta\psi/\Delta y$ and v is equal to minus $\Delta\psi/\Delta x$ and that is as far as the stream function is concerned. For potential function last time we mentioned that u we put a minus sign outside, but I pointed out that at the end it does not matter how you define your because it is relative potential function you consider that to be a negative or positive.

So, we continue to do this with a positive here in this discussion we are writing it as positive because most of the books they follow positive u as $\Delta\phi/\Delta x$ and v as $\Delta\phi/\Delta y$. And the other thing is that the velocity is u i hat plus v j hat. Now, this velocity in this case would be we write $F'(z)$ or $\frac{dF}{dz}$ that we are writing as $\frac{dF}{dz} = u(x, y) - iv(x, y)$. How we arrive at this? Because if we take the derivative of this if this is your f z if this is the expression you have for f z and then if you take a derivative of this derivative of a complex number where f itself is a complex number ϕ plus i ψ z itself is a complex number which is x plus iy. If you take the derivative it would be either $\Delta\phi$ Δx plus i $\Delta\psi$ Δx or it would be $\Delta\psi$ Δy minus i $\Delta\phi$ Δy.

So, it would be either of these. So, you can see here that here in this case $\Delta\psi/\Delta x$ is minus v and $\Delta\phi/\Delta x$ is we call this u. So, you can see that u minus iv or in other words $\Delta\psi/\Delta y$ is u and $\Delta\phi/\Delta y$ is v. So, it will also be u minus iv. So, that means, when you take a derivative of this dF /dz that would be equal to u minus iv that is a derivative of this complex potential with respect to z.

$$F'(z) = \frac{dF}{dz}$$

$$\frac{dF}{dz} = \frac{\partial\phi}{\partial x} + i\,\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} - i\,\frac{\partial\phi}{\partial y}$$

$$\Rightarrow \frac{dF}{dz} = u(x,y) - iv(x,y)$$

$$\Rightarrow \frac{dF}{dz} = \text{complex velocity referred as } w(z)$$

So, this is referred as complex velocity which is wz. So, we note here that the ww bar that would be equal to in this case square that would be u square plus v square. Because it would be u minus iv is w, w bar would be u plus iv and you take the product it would be u square minus iv whole square since i square is equal to minus 1. So, you get there u square plus v square. So, this is how you define what is called complex velocity which is $\frac{dF}{dZ}$.

What we do next is we try to write some expressions for Fz. For example, I write $F(z) = Uz$, Fz equal to some constant multiplied by z what kind of flow this signifies. We can we note here that if I take derivative of this $\frac{dF}{dZ}$ would be equal to capital U. So, capital U is a real constant I mean you could have written it instead of U you could have written it as C, C is a real constant. So, $\frac{dF}{dZ}$ would be a real constant and since we know on the other hand that complex velocity $\frac{dF}{dZ}$ is equal to $u - \hat{\imath}v$.

So, in this case I can see here on this side I have only a real component imaginary component is 0. So, then we can immediately conclude that this v is 0 here and this u can be replaced by capital U. So, then what type of flow this complex potential this Fz equal to uz signifies? Fz equal to uz signifies velocity constant velocity at in x direction. So, that means, the velocity this type of flow is given by this Fz equal to uz. So, that is the complex potential for flow along x axis.

Similarly, if you write $F(z) = -\hat{\imath}Vz$, it is a complex number. So, once again dF/ dz in this case in this case dF/ dz would be in this case $\frac{dF}{dZ}$ would be equal to $-\hat{\imath}Vz$. So, it would be it would be when you take the derivative of this becomes $-\hat{\imath}V$, then if $\frac{dF}{dZ}$ in turn is equal to $u - \hat{\imath}v$ the complex velocity. So, this implies that u is equal to 0 if I compare real with real imaginary with imaginary we see here u is equal to 0 and v is equal to capital V.

So, that means, we are talking about in this case when I say Fz equal to $-\hat{\imath}Vz$ we are talking about a flow in y direction because v is the velocity component y direction that is non-zero and whereas, u is 0 the velocity in x axis. So, this defines this type of Fz equal to $-\hat{\imath}Vz$ that defines a flow in y direction. If I say $F(z) = ce^{-i\alpha}z$. So, in that case $\omega(z)$ would be equal to $\frac{dF}{dZ}$.

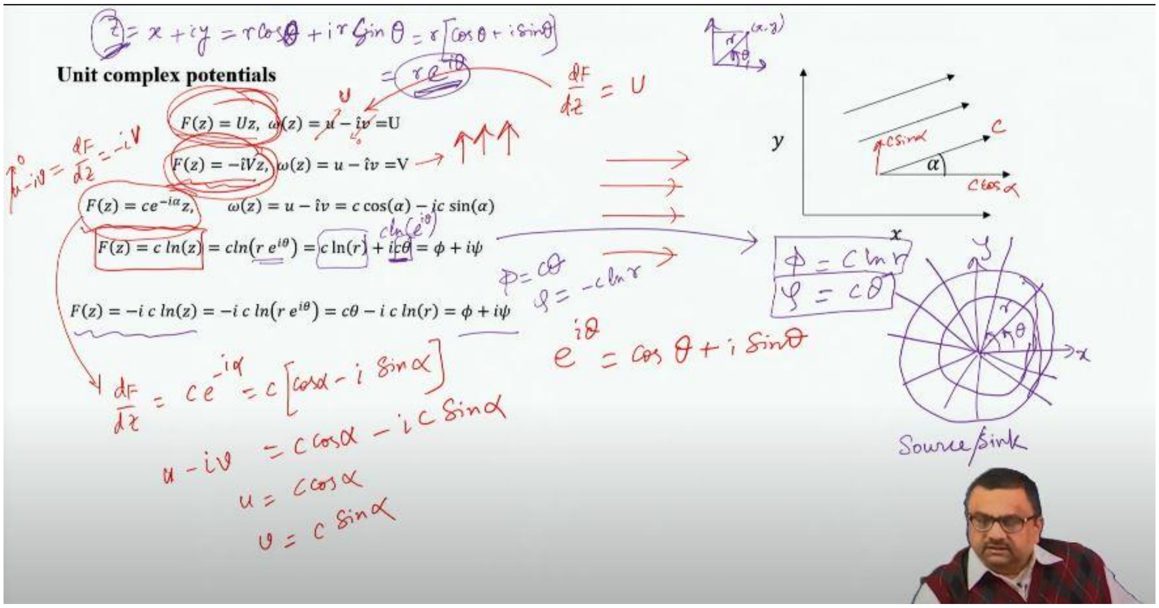
So, it would be $ce^{-i\alpha}$, right and we said this Euler theorem we talked about where. So, $\frac{dF}{dZ}$ would be equal to $ce^{-i\alpha}$ that is what. So, here $\propto$ is a constant c is a constant. So, then you can write e to the power we said that there is this theorem e to the power i θ is equal to cos θ plus i sin θ. So, then e to the power minus i∝ that would be equal to c into cos∝ minus i sin∝.

So, that is what it boils down to dF dz. So, then is equal to c cos∝ that is the real component and imaginary component is i sin∝ and on top of that this dF dz has to be equal to the complex velocity which is equal to $u - \hat{\imath}v$. So, real with real imaginary with imaginary if I match then u is equal to in this case c cos∝ and v is equal to minus and minus they are there. So, it would be c sin∝. So, what type of flow are we talking about? The flow where the x component of velocity is c cos∝ and y component of the velocity is c sin∝.

So, obviously, it is flow at an angle ∝ to the horizontal because then if this magnitude is c then the x component of the velocity would be c cos∝ and the y component of the velocity would be c sin∝. So, that is so, we are talking about an inclined flow the flow is inclined at ∝ the angle of inclination is ∝ to the x axis. So, we talked about horizontal flow vertical flow and flow at an angle. So, you can see these are referred as elementary flows elementary complex potentials. So, this is one this is one this is one then we talk about $F(z) = c\ln(z)$ that is another elementary complex potential.

$$F(z) = c\ln(z) = c\ln\left(r\,e^{i\theta}\right) = c\ln(r) + ic\theta = \phi + i\psi$$

So, what type of flow we are looking at here when we say $F(z) = c\ln(z)$. So, first of all we need to understand what type of flow is this. So, what we are doing here is $F(z) = c\ln(z)$ means what is the ln z now z is equal to x plus iy that is the definition of z or we can write what is x, x is if I go to the r θ system we write it as r cos θ + i r sin θ. Because x is r cos θ and y is r sin θ that is that is what when we go from a Cartesian system x y system to r θ system this is an angle θ and this is the distance r and the coordinates are x y.



So, we know that the x coordinate is r cos θ and y coordinate is r sin θ. So, r cos θ plus i r sin θ or we can write r cos θ plus i sin θ. So, this is taking me to again using Euler theorem it is r $e^{i\theta}$. So, instead of z I can write z I can replace z as r $e^{i\theta}$. In fact, that I am going to do interchangeably I mean when I am working with a Cartesian system I am happy with z equal to x plus iy.

Moment I want to go to the cylindrical system I will not work with z I will simply replace the z by r $e^{i\theta}$. So, here if I put this z as r $e^{i\theta}$ the advantage is I have ln outside. So, I want to make use of that. So, then immediately real and imaginary component breaks down because c ln r, ln of AB product of AB it would be ln A plus ln B. So, it would be $c\ln(r\,e^{i\theta})$.

Now, c ln r plus this is $c\ln(r\,e^{i\theta})$. So, what is this ln e to the power i θ? That would be ln of e to the power something is then that something itself. So, that is why it is i c θ or c i θ, c i θ is coming out there. So, I can see here this is completely a real component and this is completely an imaginary component. I could break it into f z into real and imaginary and what is the original definition of f z? Our original definition of f z was $\phi + i\,\psi$.

Φ was the potential function and ψ was the stream function. So, now, I am writing f z is equal to $\phi + i\,\psi$. So, the immediate offshoot of this is ϕ is equal to cln r and ψ is equal to c θ. What type of flow are we talking about? Let us say streamlines. Streamlines are the lines along which stream function is constant.

So, here ψ is equal to c θ. c θ what are the lines I mean I am talking about this is my x this is my y. So, this is x this is y this is r and this is θ. So, what type of lines will have c θ as constant? That means, what type c is anyway a real constant. So, what type of lines will have θ as constant? If I mean that I am drawing a line and all points on this line will have θ as constant.

So, this line I can see this is having a particular value of θ. Similarly, I draw another line. So, this line will have another set of another θ. So, the lines that are drawn like this they will all have θ this is for a unique value of θ, this is for another unique value of θ, this is for another unique value of θ. So, the line along which θ is constant if you try to look for line along which θ is constant those are the lines going radially from the origin.

So, that is what the lines that will represent the streamlines and the potential lines are the lines along which c l n r is constant. So, in other words the r is constant. So, what are the lines along which the r is constant? The line along which r is constant if I draw a circle. So, this line will have certain value of r that is maintained at every point on this line. Similarly, another one will have another circle you can draw putting this 0 0 as the origin of the circle or the center of the circle then you will get the get another line.

So, these potential lines are the circles here and the streamlines are the radial lines. So, what type of flow that is? We have already talked about various elemental flows. So, this is the streamlines are radially moving upward or inward depending on the sign of c if c is positive the flow is radially outward and if c is negative flow would be radially inward. So, we are talking about source or sink. So, source if the flow is radially outward sink if the flow is radially inward.

So, we have in this case $F(z) = c\ln(z)$ represents the flow of source or elemental flow representing source or sink. This is an important observation and we can club these two other elemental flows using the superposition principle. Now, if I look at this other one

$$F(z) = -i\,c\ln(z) = -i\,c\ln\left(r\,e^{i\theta}\right) = c\theta - i\,c\ln(r) = \phi + i\psi$$

So, then what do we have here in this case? We have this now in this case ϕ is c θ and ψ is c ln r. So, once again, the ψ are the streamlines. These are the lines along which the r is constant. So, line along which r is constant is the they are the circles. So, this time, these circles would be the streamlines, and the circles would be the streamlines because these are the lines along which r is constant, and the potential lines will be lines that are going radially outward. So, here we are talking about a vortex of course, an irrotational vortex.

So, this is something which we may like to make note of. So, these are so, essentially if I remove everything and then I try to only identify the complex potentials this is the elemental complex potential for flow in x direction, flow in y direction, flow at an angle source or sink vortex. So, we could identify some of these elemental flows. So, then we can play with a superposition and see where we head to. Now, this source and sink, you may note one thing here: we directly link this u to the velocity in the x direction, and there is the v with the velocity in the y direction.

But here still this c we can see this c would be essentially the velocity at an angle. So, c cos∝ means that means, c also we can replace by the magnitude of the velocity. So, that is also taken care of, but here, in case of source or sink or in case of vortex, what does c mean? And you remember we have talked about vortex and how to estimate the strength of a vortex in subjects like those we had already talked about those. So, now, we have to get some physical meaning of these c term at least in these two cases source or sink and vortex what does this how c can be related to some physically significant parameter. So, for this purpose, we need to look into the velocity that we get out of this.

So, our source or sink $F(z) = c\ln(z)$. So, what would be W z? W z is $\frac{dF}{dZ}$. So, dF dz would be derivative of ln z would be 1 by z. So, derivative $\frac{dF}{dZ}$ is $\frac{c}{z}$ and on the other hand complex velocity we have already used this few times complex velocity is u-iv and that is equal to c by z. What we are interested is we are interested to know what is u what is v, but we cannot get there because the right hand side z is x plus i y. So, we cannot we had to compare real with real imaginary with imaginary right side is all mixed up.

So, we have to split them how do we split them how do I how do I get this out from here. So, one thing is that instead of z since we know that we are talking about source or sink and since we have to work with cylindrical system cylindrical coordinate system. So, I would like to use instead of z I want to use $re^{i\theta}$. It is that same thing I may repeat one more time and this is obviously, for the last time z is equal to x plus i y and x is r cos θ plus i r sin θ and then we have this is equal to $re^{i\theta}$ using Euler theorem. So, this is $\frac{c}{re^{i\theta}}$ or you can write this as $\frac{c}{r}e^{-i\theta}$ still it does not help because I have to split in split real and imaginary.

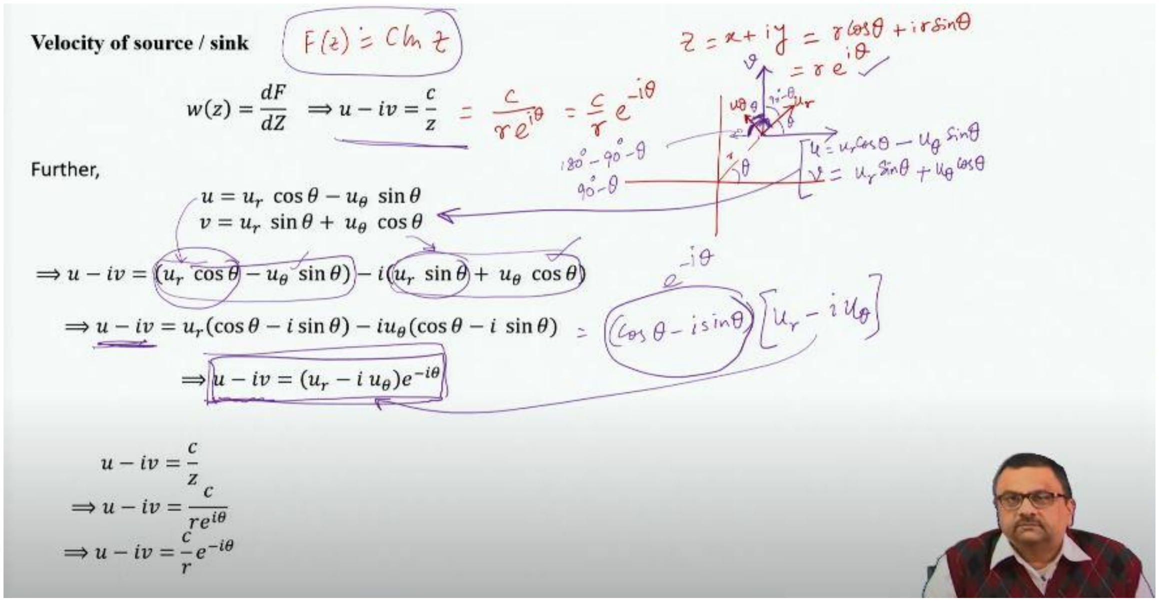
$$w(z) = \frac{dF}{dZ} \quad \Rightarrow u - iv = \frac{c}{z}$$

$$u - iv = \frac{c}{z}$$

$$\Rightarrow u - iv = \frac{c}{re^{i\theta}}$$

$$\Rightarrow u - iv = \frac{c}{r}e^{-i\theta}$$

So, that I can compare with u and v, but here I have some if I play with the left hand side because left hand side I should not be talking about the Cartesian u and v instead I need to talk about cylindrical components that is u_r and u_θ. So, how I come how I convert that. So, if I have the x y coordinate system and suppose I have let us say someone is there it is having a velocity u r and u_θ. So, let us say this is the point which is having u r this is the r direction.



So, this is r. So, this is u r and then what is u_θ? u_θ would be perpendicular to this. So, that is u θ r θ system when we work with cylindrical and Cartesian system. So, this is u r and this is u θ. So, and what is u? u is the velocity in x direction this is u and this is v.

So, now, if we try to relate this. So, we have u is equal to this angle is θ that is what our idea is. So, then this angle will also be θ. So, u would be equal to u r will have a component which is u r cos θ and u θ will have another component, but this would be in the reverse direction. So, there would be a minus sign and if this angle is θ then this angle is 90 degree.

So, this angle would be in that case 180 degree minus. So, this angle would be what? This angle would be I can see 180 degree that is the total angle minus 90 degree is taken up by this one 90 degree. So, this angle between this two vectors u r and u θ that is taken up by 90 degree minus this angle θ. So, that gives me this angle the remaining angle. So, this angle is then 180 degree minus 90 degree minus θ.

$$u = u_r \cos\theta - u_\theta \sin\theta$$

$$v = u_r \sin\theta + u_\theta \cos\theta$$

$$\Rightarrow u - iv = (u_r \cos\theta - u_\theta \sin\theta) - i(u_r \sin\theta + u_\theta \cos\theta)$$

$$\Rightarrow u - iv = u_r(\cos\theta - i\sin\theta) - iu_\theta(\cos\theta - i\sin\theta)$$

$$\Rightarrow u - iv = (u_r - i\,u_\theta)e^{-i\theta}$$

So, that is equal to 90 degree minus θ. So, this would be u θ cos of 90 degree minus θ this component would be u θ cos of 90 degree minus θ and cos of 90 degree minus θ would be sin θ. So, u is equal to u r cos θ minus u θ sin θ that is how you will get there. Whereas, now v if you write v would be equal to here you we will have this angle is θ means this angle is 90 degree minus θ. So, v would be equal to first the component of u which is u r cos of 90 degree minus θ is sin θ. So, u r sin θ that is the component for u in v direction and u θ will have similarly another component both would be positive in this case.

So, there will be a plus sign and what would be these angle in this case because this is this is 90 degree minus θ this whole angle is 90 degree. So, this has to be equal to θ. So, then this would be this angle has to be equal to θ. So, then this one would be I mean u θ the component of u θ in v direction would be u θ cos θ.

So, this would be u θ cos θ. So, exactly this is what is reflected here.

$$u = u_r \cos\theta - u_\theta \sin\theta$$

$$v = u_r \sin\theta + u_\theta \cos\theta$$

So, now, if we try to this is vector identity I mean it has nothing to do with whether it is a source or sink or a vortex. I am simply shifting from a Cartesian system of u and v to cylindrical system u r u θ. Earlier we have shifted from Cartesian system z equal to x plus i u i to cylindrical system z equal to r e to the power i θ that is all it has nothing to do with whether what elemental flow I am talking about. So, in this case now u-iv if I try to write u-iv is this whole thing is u and this whole thing is v.

$$u - iv = \left(u_r \cos\theta - u_\theta \sin\theta\right) - i\left(u_r \sin\theta + u_\theta \cos\theta\right)$$

$$\Rightarrow u - iv = u_r(\cos\theta - i\sin\theta) - iu_\theta(\cos\theta - i\sin\theta)$$

$$\Rightarrow u - iv = \left(u_r - i\,u_\theta\right)e^{-i\theta}$$

The good part of it is this e to the power i θ when we compare this u-iv with this u-iv which is $\frac{c}{r}e^{-i\theta}$, then this $e^{-i\theta}$ and this $e^{-i\theta}$ will cancel and then we can split it into real and imaginary component. One thing I want to want you to make note of is that this exercise that we have done of $u - iv$ going to $\left(u_r - i\,u_\theta\right)e^{-i\theta}$. This expression is independent of what elemental flow we are talking about this is sort of vector identity. So, you must this will be in fact, later on we will simply recall this final expression and you are expected to go through this exercise quickly and see that how it makes sense. Essentially this z to r e to the power i θ or $u - iv$ as $\left(u_r - i\,u_\theta\right)e^{-i\theta}$, these we will just take as it is I mean as if it is already proved because we have already seen that this is how it is happening.

So, we will utilize this in the next lecture what we will do is we will try to utilize this. Now, we are in a position to split u and v. So, if I split u and v and find out what is u and what is v in the case of this elemental flow C ln z. Our objective is to link this u and v whatever magnitude we get link that to C because we earlier for uniform flow instead of C we have written directly capital U instead of vertical flow instead of C we have written directly the capital V. So, here in case of source sink we have to be able to connect this C to some physically meaningful variable.

So, for that we are seeking what would be the meaningful variable obviously, the velocity or related term flow rate etcetera. So, to that effect we are trying to do this exercise. I will continue this in the next lecture on complex potential. That is all as far as this lecture module is concerned. Thank you for your attention.