Momentum Transfer in Fluids
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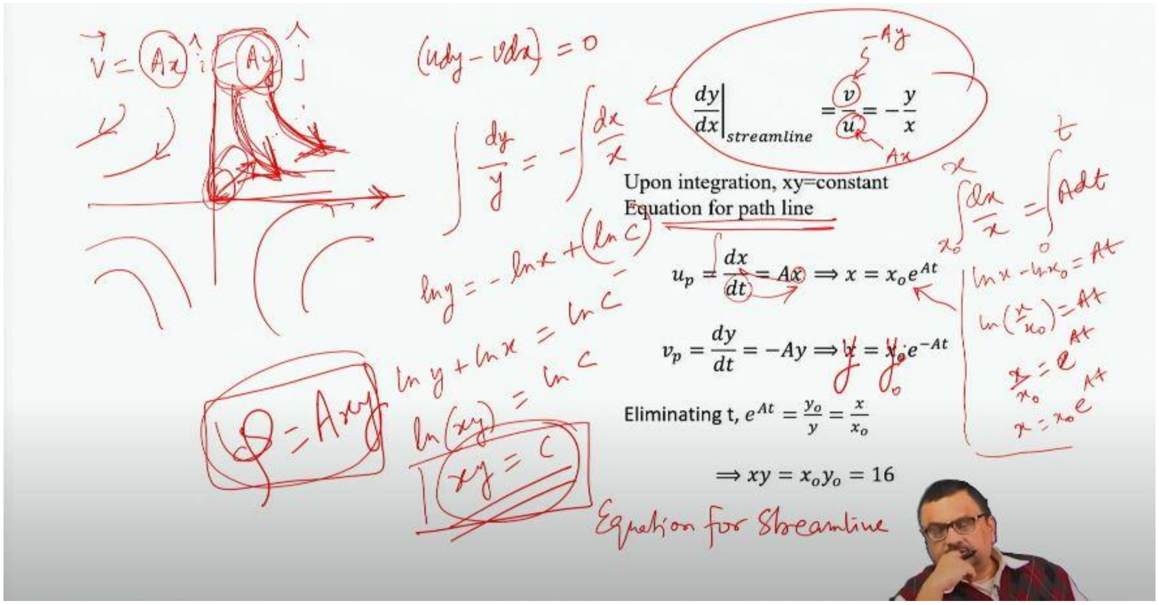
I welcome you to this lecture on Momentum Transfer in Fluids. In our last lecture we talked about stream function and potential function and how we can utilize them to get a visual of the flow process and find out the velocities at different points. In continuation to that, now we are going to define something called a complex potential, where this potential function stream function they will be combined. Before we get to that, a couple of points I need to talk about in the context of these stream function and potential function. First of all, we need not have to necessarily have the potential function for example only for the stream function. Stream function and potential function they need not have to be only for the Cartesian system that is $u = -\frac{\partial \phi}{\partial x}, \; v = -\frac{\partial \phi}{\partial y}$ that need not have to be the case.

You can have for cylindrical coordinate system also you will have the potential function and stream function values. Just an example is given here. You can see here the velocity in cylindrical coordinate system,

$$V_r = -\frac{\partial \phi}{\partial r}, \; V_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Similarly, for stream function also you will have the definition. They would be taken into account for cylindrical coordinate system as well. So, instead of u and v you can define v r and v θ in terms of stream function and potential function. So, these definitions exist and if you need to work with cylindrical coordinate system you can very well do that. You do not need to have this definition of u is equal to ∂ ψ ∂ y and v is equal to minus ∂ ψ ∂ x that need not have to be always in terms of Cartesian system. You can extend that to cylindrical system as well.

That is one thing and the other thing is when we talked about these streamlines I mean just like we said that our classical problem that we were looking into was $A_x\hat{i} - A_y\hat{j}$ and then we said that these are the lines. We can see very well that these are we said that the our condition was that u dy minus v dx that is equal to 0 that into k hat you remember this k hat that is equal to 0. So, from here we can write dy dx for along a streamline that is equal to v by u and for this A x when v hat when the velocity field is $A_x\hat{i} - A_y\hat{j}$ you can see that these values would be simply v would be replaced by minus A y and u would be replaced by A x. So, this is equal to minus of y by x. So, in this case you have if this is the if dy dx is equal to minus y by x.



So, you can write this as what you can write this as dy by y that is equal to minus dx by x. If you try to do this if from here you can write this and if you do this integration if you do this integration between some value x naught and y naught to some value x and y if you if you do this integration this is from some y naught into y and this is from x naught into x. Then you will have ln of y by y naught that is equal to minus ln of x by x naught either that or you can do you cannot you need not have to have a finite integral you can have just dy by y equal to minus dx by x. So, if you do this if you do not do this if you do this integration you can see that it is ln of you can write this as since there is a minus sign. So, this gets reversed x naught by x.

So, it becomes ln of x y by x naught y naught that is equal to 1. So, that is equal to 0. Let us look at this with the regular integration if we do the regular integration here. So, then it would be ln of y is equal to minus ln of x plus let us say ln of c this becomes a constant of integration if I write this.

So, then it would be ln of y plus ln of x that is equal to ln of c. So, ln of x y that is equal to ln of c. So, you get this x y is equal to c. So, x y is equal to c means this these are the equations these are this is the equation for streamline. So, you can call this x y is equal to c as the equation for streamline and that is exactly what you observed here in this case x y is equal to c these lines these are these are these corresponds to x y is equal to constant lines.

And on top of that you can we last time we saw that it is if we go for these 4 quadrants and if we see that in the positive quadrant v the new component is positive and the v component is negative v component is negative. So, u component is positive and v component negative. So, the streamline will have direction like this. Similarly, here in this case on the on this quadrant the u component is negative because x is negative and v component is the y component is positive. So, y component is positive means v becomes negative.

So, y positive means v becomes negative. So, this would be like this the direction would be like this because u would be u would be in negative x direction v would be in negative y direction since there is already a minus sign sitting there. So, you can find out the directions from by looking at the velocity field you can even draw the streamlines by doing this and on top of that you have the you have the you have the value of ψ that is equal to A x y. That was the equation for stream function assuming that the streamline that passes through 0 0 the origin. So, this is also another streamline.

So, this is the corresponding stream function value is 0. So, you have to find out this corresponding stream function values for each streamline and then for example, I see there is a huge gap here. Here the gap is less, but here the gap is becoming more. Essentially this is a typical flow past 90 degree bend. So, it is you can expect that if this is the direction of the flow this is the direction of the flow.

So, the flow is coming from the top here taking a bend 90^0 bend and then moving towards this way. So, you can see that these region here the velocity is least because these region is pruned to so-called I mean you can think of if you go to a corner of the room and if you with a bucket full of water if you flow some water that way you will find that it will take a turn in the corner and turn to the other side, but there is a good possibility that this region near the corner the absolutely close to the corner that region tend to be more the velocity would be less tend to be more of a stagnant place. So, that is so these are the things you can you can immediately conclude out of it. So, this is as far as the streamline equation for the streamline. Equation of the streamline if we go follow this route this is xy equal to c if we go if we follow the route of stream function ψ is equal to a xy and a xy equal to constant means it is the same thing xy is equal to constant the same line and if we ask you to find out the streamline equation for the streamline that passes through some coordinate of any coordinate you pick up there would be a streamline passing through that every bit of every point there would be a streamline passing through that it is only that you choose to draw streamlines by keeping the ψ value equi spaced so that you can make meaning out of it that is your that is you have you put together, but ideally any point through which you can you can draw a streamline.

So, if I give you those values of x and y coordinates you can find the equation for that streamline, but that will follow anyway xy is equal to constant. So, any streamline passing through let us say if I say any streamline passing through x equal to 1.5 and y equal to 7.5. So, it would be xy is equal to constant.

So, xy is equal to 1.5 into 7.5 that is the equation for the streamline the entire streamline will follow that same equation xy is equal to product of 1.5 and 7.5. So, this is what the equation for streamline is. Now, when it comes to the path line if someone wants to know the path line we said what was path line streamline had the definition is tangent to that line has to follow the direction of the velocity at that point. So, path line is truly if you put a paper boat and see where the boat is traveling. So, put some tracer and see where the tracer is going. So, if you want to find out the equation for path line then you what you will do is you will write first of all you have to take this to the Lagrangian form because you are keeping track of the those tracer particles.

So, you have to write you have to bring in the time. So, for velocity field was Eulerian. So, we have not talked about time, but now you have to write u as dx/dt is equal to Ax and v as dy/dt is equal to minus Ay you have to do the integration which will give you integration of dx would be equal to this dt will go to that side and then and this x will come to the denominator. So, it would be integration of dx by x that is equal to Adt. So, dx by x is ln of x and so this is taking you to ln of x and if at time t equal to 0.

So, if when you do this integration you may write do this integration between x naught to x and time equal to 0 to t. So, if you do this you will end up with ln x - ln x_0 that is equal to At and then you have ln of x by x_0 is equal to At and then you have x is equal to or x by x_0 is equal to e to the power At. So, x is equal to x_0 into e to the power At. So, these equations you will invariably end up with. So, that is where you get there.

Similarly, for the velocity in y direction you have dy/dt and dy/dt would be in this case Ax minus A xi at minus Ay j at. So, velocity is minus of Ay. So, that is going there and so you have the minus Ay is equal to dy dt again y comes out as dy by y dt goes there do the integration. So, these becomes equal to here is a major slip this is y is equal to y naught e to the power minus At. So, this is y is equal to y naught e to the power minus At.

So, that is what you get. So, if you know that you want to find the path line, you have to eliminate the t between these two. So, you have to eliminate t. So, you can see e to the power At can be written as x by x_0 and e to the power At is written as y by y_0 by y since it is e to the power minus At. So, from here you can get xy is equal to x_0 here you get xy is equal to x_0 y_0. So, these becomes the equation of path line.

So, you can see that the equation for streamline and equation for path line they are same. In fact, that is what is expected we said that the equation for streamline and path line will be same as long as the flow is steady state here the flow is steady state. Now, it is, but when it comes to deriving the equation for path line one has to follow the Lagrangian framework which is given which we discussed just now. And when it comes to definition of streamline one has to follow the Eulerian framework which we discussed earlier and here also a little bit. So, both are going to the same place, but through different means and when flow is unsteady the obviously, the equations will be different.

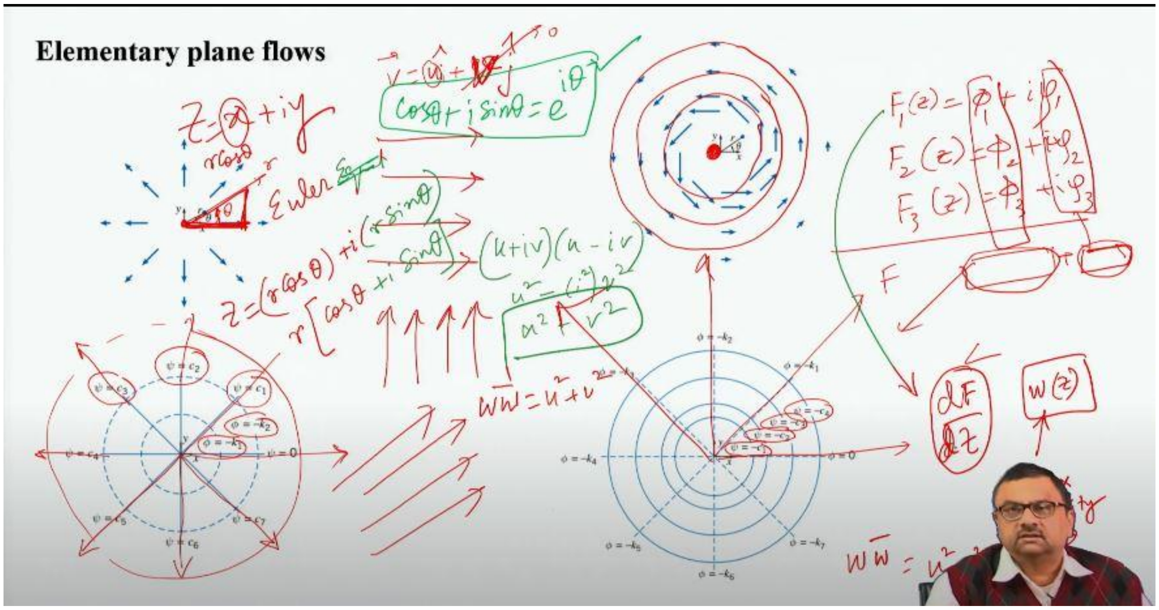
So, this is one thing. Next what we are going to do is we are going to talk about some complex flows. Complex flow in the sense where we will define complex potential as I mentioned in one of the previous lectures complex potential F as ϕ plus i ψ. So, you can see here that these complex potential first what we will define is we will define complex potential for some elementary flows. What are these elementary flows? Elementary flow can be simply flow in x direction. That means, here if you write velocity field as $u\hat{i} + v\hat{j}$ So, then here in this case u is constant and v is 0. So, we can have uniform flow in x direction. We can have uniform flow in y direction. So, in this case u is 0 plus v, but v is existing or we can have uniform flow at an angle because these are all inviscid flows. So, all fluids are moving in mass I mean there is not one layer sliding against the other.

So, we can have the flow at an angle. So, that would be in this case you will you have both u and v existing. So, this is flow at an angle or the flow could be a source. Source means from a particular point this is x direction this is y direction and then you can write r θ system also particularly when it comes to source the it will be better to work with r θ system than xy system. So, that is why you can see this is given as r and this is given as θ this is r and this angle is θ.

So, this is known as source. Source means from a point fluid is emanating out. Suppose I have a there is a flow going in and suddenly you puncture you make a hole and the fluid that comes out that would be a source or in a big pool of liquid you put a tube and then you introduce some fluid through that that would be typically a source from a source fluid will radially go outward. It could be just the other way of putting it in the sink, where the flow would be radially going inward towards a point. So, that the flow direction would be simply reversed in this case flow direction would be reversed. Now, when it comes to this since we are talking about this 2D flow the all these would be per unit depth perpendicular to the screen.

For example, we will have a strength of a source or a strength of a sink very soon we will discuss those very soon we will get to the derivation of those. There it would be again volumetric flow rate per unit depth perpendicular to the screen that is the volume that leaves from that point as a source volume that leaves per unit time per unit depth perpendicular to the screen. So, that is that is that we would call it a strength of a source similarly a sink. So, that is one thing and the other thing is vortex where flow would be taking a turn like this.

So, this is something which is shown here. So, this is known as a vortex. When it comes to a vortex we had talked about earlier that vortex would be again you can have a viscous core and inviscid periphery. So, that means, vortex means there would be. So, vortex means there would be these are the circular streamlines that are there.



So, these are the circular streamlines. So, here the streamlines are circular vortex means the streamlines they are circular and then you can apply these complex potential and everything for the treatment that we have for inviscid flow for these vortex except near the core where the rigid body motion that those equations they come into play we discussed these earlier where one layer would be sliding against the other and the flow is no longer irrotational. So, leaving a very small core of few millimeters leaving that core outside it would be all streamlines it would be simply circular. Earlier we said that the flow is in circular line now we call that circular line to be streamline. And then we have. So, now, if we try to find out the streamlines for source and source or sink and if we try to find out the streamline for vortex here we can see these are the streamlines ψ is equal to c 1 ψ is equal to c 1 ψ is equal to c 2 ψ is equal to c 3 like this.

So, these are the streamlines ψ is equal to 0 you have to choose some 0 stream that is your choice I mean you could have taken ψ is equal to 0 as this line, but you will choose in a way such that it helps in your calculation. So, these are the streamlines. So, these are the streamlines because this is a source. So, this is radially coming outward. So, that makes perfect sense that if it is a source the streamlines would look like this lines that are radially coming outward.

So, how the potential lines look like potential lines would be path is orthogonal to the stream lines. So, that means, if it is a radial line orthogonal to that at every point. So, that would be a circle. So, this circle gives me the potential line this the other circle gives me another potential line. So, you can see here ϕ is equal to $-k_1$, $-k_2$, they put a minus sign because this we have this $\nabla\phi\ \partial$ x and we u as minus $\nabla\phi\ \partial$ x etcetera.

If we are using positive sign then we ϕ equal to k_1, k_2. So, you can have another line these are all potential lines because flow will take place perpendicular to the potential lines that is the definition of potential line. Whereas, when it comes to vortex we will have how are the streamlines vortex means circular lines. So, these circular lines are streamlines now. So, you can see ψ is equal to c 1 ψ is equal to minus c 1 ψ minus c 3 ψ equal to minus c 4.

So, these are the streamlines and the potential lines now it has to be perpendicular to the streamlines. So, potential lines would be the radial lines phi is equal to 0 phi equal to minus k 1 k 2 k 3 like this. So, these are the potential lines. So, it is it just got reversed when you have a vortex streamlines are circular potential lines are radial lines. Whereas, when it comes to source and sink the streamlines are radial radially outward or inward lines depending on whether you have source or sink, but the potential lines are circles.

So, these are called elementary flows. Now, we will have there are other elementary flows as well we will discuss this down the line, but these are most basic elementary flows I would say. What we will have is we will define complex potential for these elementary flows. That means, we will define what is called $F(z) = \phi + i\psi$ for these elementary flows. And then the idea is that when we superpose one with the other that means, when I have a vortex and on top of that I have a source and on top of that I have a uniform flow all three happening simultaneously in one place. So, can you tell me how the streamlines look like? What would be the velocity at different points where the streamlines are coming close to each other? So, for that what we need to do is we have to write F_1 for source, F_2 for uniform flow, F_3 for vortex, all three are happening simultaneously.

Sum them up and whatever the net value ϕ 1 plus ϕ 2 plus ϕ 3 whatever net value of potential function I get and net value of stream function I get. So, these would be the complex potential that I get by summing them that would be the resultant complex potential. So, this would be the resultant potential function and this would be the resultant stream function. And this stream function once I have this stream function then we can get quickly the profiles of streamlines. Once we have these we can get the profiles for potential lines.

And then we can extract from there how where the flow is increasing flow is decreasing what are the velocities at different points and all this. And once we have velocities in place then we can apply Bernoulli's equation. Bernoulli's equation is applicable. I mean that we will discuss down the line that there is one precondition for applying Bernoulli's equation, which is that the flow has to be irrotational, that is, the curl of the velocity field has to be 0. So, we can apply Bernoulli's equation on that velocity, etcetera, and find out what the pressure would look like and all these. So, that is the overall idea when we talk about complex potential and when we discuss when we discuss these elementary plane flows that is the idea that we want to we can superpose as and when necessary.

And it could very well be that your that the source is located here at this point maybe at the origin, but the vortex is located little bit away and uniform flow is at an angle. So, if they are, they may not have to be on the same 0 0 here. If we put x equal to 0 and y equal to 0, all of them need not have to be on x equal to 0, y equal to 0. One may be shifted and then we want to see the combined effect. So, we have to do some adjustment to make sure that these two we have to make some adjustment to make sure that these shift of coordinate is taken care of. So, in that picture, we have a source acting there, a sink acting there, a vortex acting there, and some other flow we will discuss doublets and all.

So, they are acting there. So, now, I have a combination of all these they are not at the same point a little bit shifted and now over this entire spectrum now tell me how the streamlines will look like how the what are the values of stream functions for these streamlines. So, these are these are some of the things that that we are we are interested in that is what we are heading to. And one thing is there that then moment we have this superposition we of course, we get summation of ϕ and summation of ψ, but these directly from these F1 z we can we can get dF dz which is which is which will give you something called we call this Wz that is that gives me what you call the complex velocity. Complex velocity which is given as u minus iv u is the velocity in x direction v is the velocity in y direction. So, W and complementary of W that gives me the u square plus v square right u square plus v square that is the square of the magnitude of the velocity because i square i is a complex number mind it.

So, if i is a complex number. So, then u minus iv. So, that means, so, w w bar would be u plus iv into u minus iv. So, that means, equal to u square minus i square v square and i square equal to minus 1. So, that is why you get u square plus v square. So, this u square plus v square that is that is w w bar. So, that is the magnitude that gives me the magnitude of the velocity u square resultant velocity u v are the components in x and y direction.

So, we will moment we define a complex potential we will link to this complex velocity. Now, mind it when you take a derivative we say dF dz now z itself z is what z is x plus iy in this context. So, x plus iy. So, it is important that you must know that x and y we are working with instead of that we work with a complex number which is x plus iy. So, there are two things which I before we go to the next class maybe you can do some ground work on that one is how do I take a derivative of a complex number like this where F itself is a complex number z itself is a complex number how do I take a derivative of this there is a certain theory to it.

So, you please try to go through it before coming to the next class that is one thing. Another thing is something called Euler equation which is you can see here x always can be written as r cos θ. If this is x this is r and this angle is θ. So, at all the time I can write x is r cos θ and y is r sin θ. So, z can always be written as $z = (r \cos \theta) + i (r \sin \theta)$.

So, we can write r cos θ plus i sin θ and there is this Euler equation which says that cos θ plus i sin θ that can be written as e to the power i θ. This equation also you need to see the background of this equation because this equation we will be using very often in the next lecture one is this. This is referred as Euler theorem rather. So, that is what you need to look into another is how to take differentiation of a complex number. So, these two we would be I will assume that these two are known to us as we proceed.

That is all as far as my present lecture module is concerned. Thank you for your attention. We will continue this lecture.