Momentum Transfer in Fluids
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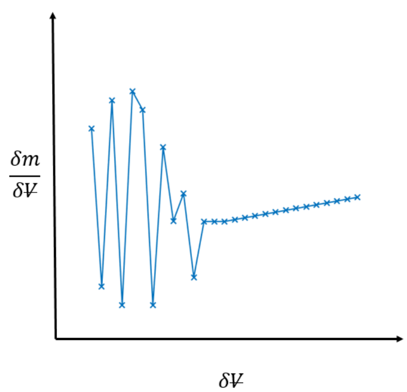
Week-01
Lecture-03

I welcome you to this lecture on Momentum Transfer in Fluids. In our last lecture, we discussed some elementary frameworks, particularly the difference between Eulerian and Lagrangian formulations.

I mentioned that we will be discussing about the Eulerian framework in this course. In this context, I talked about the probe volume. So, what we have already understood is what is the sampling volume you work with that gives you these. I mean, I am measuring density, and I am getting a consistent density, but when I reduce the probe volume I see there are fluctuations in that density.



So, as a matter of fact, if you look at this part, here also density is not exactly constant; density is increasing or it may happen that the density is decreasing. So, why would you see that there is a slight increase, or a slight decrease in density? Why would you expect this? I mean this part I can understand. Here, the probe volume is too small, number of molecules entering and number of molecules leaving at a particular time that is fluctuating because these number of molecules entering and leaving is a large fraction compared to the sitting on the in house molecules that are sitting in that probe volume. But when you are talking about a larger probe volume, there also I see a continuous increase.

The reason could be that when the probe volume increases, let us say I have a probe volume of the size of this room and another probe volume of the size of this building. So, obviously, you would see that when it comes to density of air, which I am trying to measure when I have a probe volume of the size of this building. So, I would expect the average of the density across the entire floors of this building; that is what I am measuring. So, I expect that density to go down as I go up to the atmosphere. Air density is highest near the earth surface. As I go up, I expect that the density to decrease, and at one point there is no air present. It is all vacuum, it is all space that is what we call space technologies.

So, here, you can expect that there is a continuous variation in density. Similarly, suppose I am measuring pressure. So, there is every possibility there is a flow taking place. I would expect that the pressure will decrease in the direction of the flow. So, there would be a continuous variation in pressure; I mean that is not due to fluctuations in number of molecules at a particular time entering or leaving inside in the probe volume that is a different dynamics altogether. So, I mean it is going back to the same thing. I mean, the pressure itself is a manifestation of the molecules colliding with each other and colliding with the wall, and the other movements the molecules are having.

So, this part of the curve is referred to as the implications or the immediate result of the continuum hypothesis. I mean, you are treating this part as a continuum. I mean, here you are, treating the fluid as a continuum. The molecular level fluctuations are not coming into play. Whereas, this part here is the number of the probe volume is too small, and the molecular level fluctuations are getting instrumental in defining the density.

So, I mean I can give another quick example here. let us say I flip a coin and try I am trying to find out the number of heads and number of tails. You flip the coin 10 times, and by probability, you should expect 5 heads and 5 tails, but that will never happen. You will have either 7 heads or 3 tails or it may be 2 heads and 8 tails. So, that is very much possible.

If you flip the coin 1000 times and see how many heads you get and how many tails you get, you will find that it is close to 490 maybe and 510 or 480 and 520. So, if you look at the probability or if you look at the number of times you got head or fraction of the total number of times you got head ideally, you should get that as 0.5, but you will achieve 0.5 when your sample size is large when you flip the coin 1000 times when you can leave the coil 10,000 times you will find you are getting to 0.4999 I mean you are very close to 0.5.

Whereas, if you flip the coin 10 times you will get at one time 0.2 times head, or you may get 0.7 times head that is fluctuating. So, it is because your sample size is too small. Here also, it is a completely different physical problem, but probably the bottom line is same. I mean, your sample size is small, so you get a lot of fluctuations.

Now, the question is when it comes to the Eulerian framework, where, I mean, should be our probe volume. We said that I work with a differential volume I mentioned you may recall that we had talked about a differential element, and then we said that we will say that average when it comes to density that means, mass of total number of particles divided by the volume and we call that the density at that point. And we will assume here in this case that the density is a continuous function in space and time. That means when it comes to the next differential element, then I will have again at the center of it I will have another density, I have another density, here I have another density, and this density to this density I will assume that the density is changing like a continuous function. So, between one point to the other it is not a discrete change. So, I will now assume density as a continuous function in space and time, and what is this density at a point it is the average density or total mass of the number of molecules present in the differential volume divided by the volume itself.

So, similarly, we will define velocity as a field variable in the Eulerian framework. Velocity will be a field variable; pressure will be a field variable; density will be a field variable like this. So, this entire process is referred as the continuum approach. We are assuming pressure to be a continuous function of space and time. What is the pressure at a point? Pressure at a point is average pressure as far as that differential element is concerned. And from that differential element to the next differential element, pressure is changing continuously, not discretely.

That means, here I have a pressure between these two points pressure is 0 and we have a continuous line. So, now the question is, what should this differential volume be? So, our understanding is that the differential volume is greater than this threshold. So, if the differential volume that I choose is less than this threshold, then I would have these fluctuations, and I cannot handle this then my continuum assumption is not valid. So, differential volume dimension is greater than this, but the differential volume dimension would be less than certain value so that I can treat this density to be constant within that differential element. So, with continuum assumption we can now write density as a field variable. So, density as a field variable is probably what we write density as a field variable.

$$\rho \equiv \lim_{\delta\forall \to \delta\forall'} \frac{\delta m}{\delta\forall}$$

V is struck through here this means we are talking about the volume when we write v without striking through that we are referring here as velocity. So, it has to be near this threshold. So, then we can call the density is a field variable this is how we define.

$$\rho = \rho(x, y, z, t)$$

It is not that only at that point it is valid, and next point it is not it is a continuous function. The same thing applies to the velocity, and then this velocity field would be now we break it up into unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. So, we can have a velocity u is the velocity in x direction, v is the velocity in y direction, and w is the velocity in k direction.

So, now we will be working with this Eulerian framework our differential volume that we choose. We make sure that it is sufficiently large than the issue of probe volume does not interfere. Then the question is that when this continuum approach fails, when do we I mean we must have studied in your high school days you heard of some energy equations, Bernoulli's equations, you might have some introduction.

So, there, you might have encountered something called velocity and pressure, etc. So, those are all essentially this field variable that is based on the Eulerian framework. So, then the question is where you think this will not work I mean when we have to resort to Lagrangian I mean if we have been doing all these things with Eulerian framework where is the scope for Lagrangian framework. The Lagrangian framework one has to resort to when the dimension of the conduit that means, you are having a flow through a pipe flow through a tube flow through a channel. So, there the dimension of that channel or the diameter of the tube if that is too small compared to the mean free path of molecules then one cannot operate on Eulerian framework you understand what I am saying this is the differential volume we are talking about.

And I said there is a restriction that the size of the differential volume has to be greater than some threshold. Now, if it turns out that the channel through which the flow is taking place that channel dimension is such that you cannot have a probe volume of that dimension. For example, you are having a channel dimension that is on the order of let us say nanometer. And the fluid that is flowing is essentially gas at a low pressure high temperature for which the mean free path is very large.

So, in that case you will find that this Eulerian approach does not work. In that case, you have to work with the individual molecules and their collisions within themselves. Of course, there are certain hybrid approaches possible where partly Eulerian and partly Lagrangian approaches are there. But essentially the take home message is that when the dimension of this channel is very small and when the gas is flowing or fluid that is flowing is highly rarefied that mean free path is very large. Then this type of analysis, i.e. Eulerian framework, does not work. Typically what people refer here is something called a Knudsen number, which is λ/D where λ is the mean free path and D is the characteristic dimension of the channel.

For example, it could be diameter when it comes to a capillary. So, if this Knudsen number is greater than 1 Knudsen number is large. So, in that case, λ becomes more than the characteristic dimension of the channel. In that case, limitations in Eulerian framework will come into play at that time. Now, once we have this velocity field then we will talk about the steady flow and unsteady flow. That means this velocity, we said, is a function of x, y, z, and time.

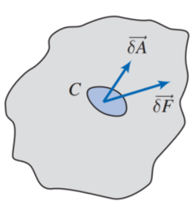
Now, suppose with time the velocity does not change, velocity is changing only with space or in some other case you will find that the velocity is changing with time only, but in space that is remaining constant. So, depending on these one can differentiate as steady flow or unsteady flow. Steady flow means with time the velocity remains constant. Velocity can vary with space. Let us say when it comes to a waterfall, you will find that the velocity increases as the water comes down.

So, velocity is not uniform, velocity is non-uniform, velocity is less up there, my velocity is increasing, or there could be similar situations, but that could be that with time, it is not changing. So, then you will call it a steady flow, and with time if it changes you call it unsteady flow, and if the flow is with space it is remaining same then it will call it uniform flow. And of course, velocity is a function of we said x, y, and z, three dimensions. It could be that it could be a two-dimensional problem or one-dimensional problem. So, in the other direction velocity is uniform.

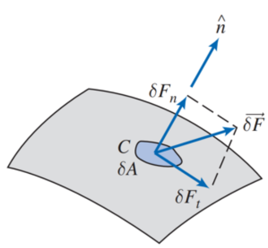
So, generally, then we consider the velocity to be at, and we treat that as a 2D problem or 1D problem, assuming that in the third dimension, the velocity is the same everywhere. So, we can work only with per unit depth perpendicular to the screen. Let us say the third dimension is perpendicular to the screen. So, we can operate. So, in many cases, you will find that our 2D or 1D flow, if we work with that, still works.

STRESS FIELD

Contact force applied to the fluid (flowing) on one side of any surface by the other



Portion of a surface in the neighborhood surrounding point C of size $\boldsymbol{\delta A}$ on which the force $\boldsymbol{\delta \vec{F}}$ acts



$\hat{n}$ is the unit vector outwardly down normal with respect to material, acted upon

In orthogonal **x-y-z** co-ordinates

$$\boldsymbol{\sigma_{xx}} = \lim_{\delta A_x \to 0} \frac{\boldsymbol{\delta F_x}}{\boldsymbol{\delta A_x}}$$

$$\boldsymbol{\tau_{xy}} = \lim_{\delta A_x \to 0} \frac{\boldsymbol{\delta F_y}}{\boldsymbol{\delta A_x}}$$

$$\boldsymbol{\tau_{xz}} = \lim_{\delta A_x \to 0} \frac{\boldsymbol{\delta F_y}}{\boldsymbol{\delta A_x}}$$

First subscript: Plane on which the stress acts (given by the normal to the plane)

Second subscript: Direction in which the stress acts

Normal stress

$$\boldsymbol{\sigma_n} = \lim_{\delta A_n \to 0} \frac{\boldsymbol{\delta F_n}}{\boldsymbol{\delta A_n}}$$

Shear stress

$$\boldsymbol{\tau_n} = \lim_{\delta A_n \to 0} \frac{\boldsymbol{\delta F_t}}{\boldsymbol{\delta A_n}}$$

The next topic that I will be discussing is something called a stress field. Let us say I have a fluid flowing on one side of any surface, I mean I have a fluid surface and then I have let us say at point C I have a force acting on that surface which is given by $\delta\vec{F}$. So, force that is given by delta F, and this is the point C on which this $\delta\vec{F}$ force acts. Now, point C here I have it has an area of magnitude delta A, and the area is a vector. So, it has a direction which is perpendicular to the surface which is given that the area vector is given here as δA.

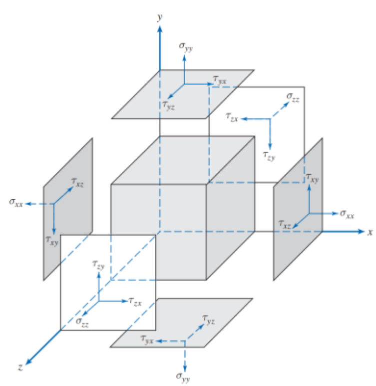
So, δA is the area of that element that we are talking about, and then this has a magnitude of δA and a direction that is normal to this surface. So, let us say this $\delta\vec{F}$ is now broken into two components: one is the tangential component, and the other is the normal component. So, this is the normal direction. So, along the normal direction the velocity is δF_n, and the tangential direction let us say this is δF_t. So $\delta\vec{F}$ is broken down into two components one is δF_t another is δF_n and then what we have here is we write the normal stress

$$\sigma_n = \lim_{\delta A_n \to 0} \frac{\delta F_n}{\delta A_n}$$

The shear stress

$$\tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_t}{\delta A_n}$$

So, this is called shear stress δF_t divided by the area that is called shear stress δF_n divided by the area that is called normal stress. Now, instead of this tangential and normal components if you have if you call it the x, y, z direction if we put it then, in that case, we have in an orthogonal x, y, z coordinates we will haveσ_{xx}, $\tau_{xy,}$ and τ_{xz}.



Stress Tensor

$$\tau = \begin{pmatrix} \boldsymbol{\sigma_{xx}} & \boldsymbol{\tau_{xy}} & \boldsymbol{\tau_{xz}} \\ \boldsymbol{\tau_{yx}} & \boldsymbol{\sigma_{yy}} & \boldsymbol{\tau_{yz}} \\ \boldsymbol{\tau_{zx}} & \boldsymbol{\tau_{zy}} & \boldsymbol{\sigma_{zz}} \end{pmatrix}$$

$$P = -\frac{1}{3}[\boldsymbol{\sigma_{xx}} + \boldsymbol{\sigma_{yy}} + \boldsymbol{\sigma_{zz}}]$$

Let us say I have the dimensions written here you can see this is a differential element this is the differential element. So, this is the differential element I have. So, let us say I talk about this surface this face. So, this face this is the x direction is the x direction. So, as far as this face is concerned, I have normal stress that is acting here over this area is σ, and the tangential stress that is acting is τ in this direction, and τ is in this direction. So, that is what is reflected here. Now there is certain catch to this subscript you can see there are two subscripts here the first subscript is the one I have two things to talk about one is the direction in which this force is acting and the other is the surface on which the force is acting and any surface that means, that surface it has it is an area vector. So, when it comes to the direction of that, I mean any surface is characterized by the direction of that area vector, and the direction of the area vector is normal to the area down which it is applied.

That means, when it comes to this area the direction as far as this face is concerned is normal to that means, that direction is in the x direction, and the force that is acting is also in the x direction. So, that is why the subscript here is x x whereas, here the force that is acting is in y direction and the area on which this force is acting. So, area on which it is acting that area also has a direction and that is the x direction, area direction of an area is normal to the area that that is what that is how area vector is. Similarly, this one here is the z direction. So, the subscript z that is showing the direction in which this force is acting and this subscript x indicates the area on which this force is acting. So, that means, I have two subscripts here first subscript represents the area on which this force is acting and the second subscript indicates the direction in on which this force is acting.

So, everything is falling in place here it is x x because it is acting on the force is acting in x direction force is acting on an area the area itself has the unit area vector is in x direction. Here also area vector is in x direction here also area vector is in x direction. However, here the direction is in y here the direction is in z. So, this is what we can do

$$\sigma_{xx} = \lim_{\delta A_x \to 0} \frac{\delta F_x}{\delta A_x}$$

Here in this case when we talk about the three dimensional here we will call this as δA_x because this area has a direction which is in x direction. This area that means, this area is delta Ay because this has the direction in y direction and similarly you have this as the z direction this area has the z direction because perpendicular to this area is in z direction.

So, this is exactly what we defined here σ_{xx}, τ_{xy}, and τ_{xz} as far as only this plane is concerned we can have here the first subscript is plane on which the stress acts first subscript is plane on which the stress axis given by the normal to the plane and second subscript is direction on which the stress axis. So, this is not right. So, direction on which the stress axis. So, that is the second subscript. So, this is how we can have a general nomenclature.

So, now, here is something called a stress tensor we define because if we have that entire the last one that we showed this is only for the area A_x. So, you can see it is all everywhere it is $A_x \lim_{\delta A_x \to 0}$. When we work with all Ax Ay and Az. So, we will have you can see σ_{xx} coming in σ_{yy} coming in. So, σ_{xx} is here σ_{xx} is there σ_{yy}, σ_{zz}, and σ_{zz}.

So, these are so called normal stresses and I will have shear stresses which are I give by box. So, these are all shear stresses. So, these are all shear stresses. So, now, as far as the total stress environment is concerned at a this is at a particular point. At a particular point we have drawn a differential element. We made sure that the probe volume is greater than the delta v prime.

So, that we can apply continuum assumption we can apply Eulerian approach. So, now, at a point momentum transfer is taking place we have to define something called a stress field and this is known as the stress tensor. So, typically, this is referred as generally whenever you have a stress tensor whatever form you do typically you put three lines on top just to make sure that it is a tensor. It is an extended form of vector you have you are all familiar with velocity as a vector, velocity is having ui hat plus vj hat plus wk hat. So, this is you have the velocity as a vector. This is an extended form of a vector, which is referred to as tensor.

In fact, you can call a vector as a first order tensor you can call a scalar as a zero order tensor and this is tensor. So, basically they all come under the umbrella of tensor. So, now, you when it comes to pressure at a point. So, what is the pressure at a point? You see here the pressure at a point is average of first of all σ_{xx}, σ_{yy}, and σ_{zz} that is acting there. So, when it comes to pressure at a point it would be the normal stresses, but you have to take average of this σ_{xx} in x direction σ_{yy} in y direction σ_{zz} in z direction.

So, one has to take the average of all these and another point one has to note is that the σ_{xx} is acting outward from the center of this differential element σ_{yy} is acting outward, but pressure is typically defined as pressure acting towards that point that is how the pressure is defined. So, we have put a minus sign outside. So, this is σ_{xx}, σ_{yy}, σ_{zz} the normal stresses average of that with a minus sign on that that is what is pressure. In fact, there is I mean whether pressure is a scalar or a vector this because pressure you would define as force per unit area essentially pressure is average of these σ values σ 's are in that means, the diagonal elements of this stress tensor.

So, that is how the pressure is defined. And so, in the Eulerian framework we will talk about pressure as the average of these normal stresses and so, pressure and the shear stress they have lot of other implications on these on this differential element you may you may note already that if the shear stresses are there they tend to deform the differential element they tend differential element tend to rotate tend to get deformed. So, those the so, shear stress will get into those. So, there would be some amount of rotationality involved in a differential element whereas, pressure alone which is just simply the average of these normal stresses can operate and they may not rotate the rotate the differential element. So, that is that that is quite apparent from this figure.

I will continue this discussion in the next lecture. Thank you very much for your for your attention. We will build on it further.