Momentum Transfer in Fluids
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Good morning. We are going to start a new topic this morning. So, far we were dealing with differential analysis of fluid motion. And in order to do that we have used extensively used Navier-Stokes equation. And there we have seen that how a governing equation, governing differential equation can be obtained from by the right choice of the Navier or right choice of the component of the Navier-Stokes equation which is relevant for the specific situation. Now, once we integrated that they are solved that equation and use the proper boundary conditions, what we obtained is the point value of the or point value of the velocity.

So, every point at the in the flow field the velocity at every point can be known, but sometimes specially in engineering applications we do not need to know the velocity at every point. Rather we would like, we would be satisfied if we get a behaviour the based on the average velocity which we can measure. So, that leads to the analysis of fluid motion using an integral approach. So, once again we are going to get the velocity and associated parameters for example, what is the force exerted by a fluid when it flows through a bend in a pipeline.

So, what is the force that is to be exerted by an external agency to keep our hose in place when water is coming out of the nozzle at a very high speed. All these engineering applications do not require the knowledge of the velocity at every point. Rather we are more interested in the all calculations based on average velocity. So, that is why then integral approach contrary to the differential approach which we have done before, an integral approach is necessary. So, we are going to start first with the basic fundamental equations of integral approach the integral equation.

So, to say without deriving them, but discussing them and heuristically identify what is the origin, what is the significance of each of these terms and how we can use it for our practical problems. So, let us begin with our study of the basic equations in integral form and the first equation the and I would like to emphasize that this portion is taken from the textbook of Fox and McDonald chapter 4. So, you get the detailed in more detailed information and a number of solved and unsolved problems in Fox and McDonald which we can use for your own practice and to get more ideas insights into these integral equations. So, for the first one we would talk about is the basic equations in integral form and then those equations are to be used for conservation of mass and for conservation of momentum. So, the you since I am talking about conservation of mass.

So, the equation in integral form that we are going to get would be nothing, but the equation of continuity in integral form. And secondly, since we are talking about conservation of momentum the equation that we get out of that the out of the conservation of momentum would be similar to the Navier Stokes equation which we have used. So, conceptually things remain same except that we will be dealing with average values, integral values not with point values. So, let us start with the basic equations in for this situation. Now, before we go into that we have to define two things, one is an extensive property and the other is the corresponding intensive property.

Now, you all know from your study of basic thermodynamics that an extensive property is something which depends on the mass. So, extensive property can change based on the amount of amount of the material which is present. So, the if n is the extensive property, then eta what you see over here this eta is the corresponding intensive property. So, eta is nothing, but the extensive property n per unit mass and I am going to expand d m which is the mass as rho times d v this v is the volume. So, the d m the differential mass can be expressed as rho times d v where rho is the density and v is the volume.

N = Arbitrary extensive property of a system

$$N\Big|_{SYSEM} = \int_{Mass\,(System)} \eta\, dm = \int_{\cancel{V}\,(System)} \eta \rho\, d\cancel{V}$$

η = Corresponding intensive property, extensive property/mass

And this integration is done on the total mass of the system when we express it in terms of d m or it can be done over the total volume of the system when we express this in terms of v the differential d v the differential volume. So, eta as I mentioned is the corresponding intensive property. So, it is extensive property per unit mass. So, eta is simply going to be eta is simply going to is defined as n by m where n is the extensive property and m is the mass of the system. So, that is the relation between eta and m.

Now, we will we will talk about the system derivatives and the control volume formulation. Once again, I will talk about the equation in general identifying and clarifying what each term means, but not the derivation. So, let us talk about a system in which there is an extensive property n which keeps on changing with time. So, the left-hand side that you see over here is the time rate of change of extensive property in the system. This n could be anything it could be mass; it could be momentum and so on.

Now, let us think about mass because it is easier to comprehend based on our understanding of something that comes in. So, some mass coming in some mass going out and so on. So, the time rate of change of an extensive property in a system is going to be equal to there is going to be one transient term and if you look at this eta times rho dv. So, rho dv is nothing, but mass. So, this is whatever this is essentially whatever be the mass inside the control volume.

$$\frac{\partial N}{\partial t}\Bigg|_{SYSEM} = \frac{\partial}{\partial t}\int_{CV} \eta \rho d\cancel{V} + \int_{CS} \eta \rho \vec{V}.d\vec{A}$$

So, this is time rate of change of the of the of the extensive property inside the control volume. The next comes this integration over the control surface, this CS refers to control surface. So, what is a control surface? As I have explained a control surface is an area which does not have any mass, but these control surfaces are used to define a control volume. The control surfaces are permeable to the intensive property or the extensive property. So, when again once again if you think about mass.

So, mass can enter through this control surface and as it enters and leaves through any other control surface. So, there are multiple control surfaces let us say that that is that are used to define a control volume. So, through one surface or through a number of surfaces mass is coming in through a number of such surfaces control surfaces mass is going out. So, if that is the case then the algebraic sum of all mass added and subtracted essentially gives me the amount, net amount of the extensive property in this case mass which is entering the control volume. So, the mass of a control volume may change as a result of all the masses that are coming and leaving which is the change is expressed by the first term on the right-hand side.

$\left.\frac{\partial N}{\partial t}\right|_{SYSEM}$ = Total rate of change of any arbitrary extensive property of the system

$\frac{\partial}{\partial t}\int_{CV} \eta \rho d\forall$ = Time rate of change of the arbitrary extensive property within the CV

$\int_{CS} \eta \rho \vec{V}.d\vec{A}$ = Net rate of efflux of the extensive property, N, through the control surface

Now, the control volume is situated inside the system. So, if that mass inside the control volume changes then obviously, that extensive property in the system will also change. So, once again the if I look at the significance of each of these terms dN dt system the left-hand side is the total rate of change of any arbitrary extensive property. I have given you the example of mass, but it could be anything of the system. So, system and control volume are two different things.

The room that I am sitting in if I consider that as a system and this bottle as my control volume then the control volume and the system are clear. Let us say this the same amount of water is coming into the bottle from the room and some amount is going out of the bottle through the through another control surface. If that happens the total amount of water contained in this bottle will change with time if the inflow and outflow rates are not the same. Now, since the total mass of water present in this room is constant. So, therefore, whatever change that you are having here inside the control volume as a result of all these inflow and outflow terms must be equal to the time rate of change of water present in the room which is the system.

So, the system and the control volume the control volume is within the system. Any change in the control volume is going to cause a change of that extensive property in the system as well. So, the first term on the left on the right-hand side is time rate of change of the arbitrary property arbitrary extensive property within the control volume and I have already explained the integration over control surface of eta rho v dA this is the net rate of efflux. Efflux is the word which signifies inflow and outflow taken together. So, efflux is the net rate of inflow and outflow of the extensive property m through the control surfaces.

So, that is the significance of the second term on the right-hand side. So, once again the system the control volume the control surfaces the time rate of change of extensive property inside the control volume inside the system and the efflux of the extensive property through the control surfaces all are related by the equation that is shown at the top. Now, let us try to apply this equation and try to see if we can get some equations which would be of use to us. So, in order to do that the first thing that we are going to do is we are trying to apply the conservation of mass. So, when my extensive property n is mass eta which is the intensive property intensive property defined as extensive property per unit mass.

Conservation of Mass N = Mass, $\eta = 1$

$$\left.\frac{\partial N}{\partial t}\right|_{SYSEM} = \frac{\partial}{\partial t}\int_{CV} \eta \rho d\cancel{V} + \int_{CS} \eta \rho \vec{V}.d\vec{A}$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho d\cancel{V} + \int_{CS} \rho \vec{V}.d\vec{A}$$

So, obviously, my value of eta is simply going to be equal to 1. Now, I am I have my basic governing equation as this and in the special case when the control volume and the system coincide this mass of the system cannot change that is essentially conservation of mass. So, if the mass of the mass of the system cannot change and for the case where the control volume and system are identical for that case the left-hand side is simply going to be equal to 0 that is essentially a statement of the conservation of mass. So, what I have on the right-hand side is del del T of rho dv plus CS integration over the control surface rho v dA. Now, this v is relative is for a stationary coordinate system.

So, incompressible if it is an incompressible fluid then we are and the size of the control volume is fixed then the first term on the right-hand side this term is going to be 0. If it is an incompressible fluid and the size of the control volume is fixed then what we have is this equation that is something which we have used or you definitely have used it in the past which is the conservation of mass equation expressed conservation of mass equation expressed in an integral form. So, what you get is if you if I have 5 such surfaces then essentially what I am saying is that rho 1 A 1 plus rho 2 v 2 plus rho 3 v 3 and so on up to rho 5 v 5 is equal to 0 that is the conservation of mass for an incompressible fluid. Now, I have put this plus minus sign over here and the this one here is the magnitude of rho v A n. What does that mean? When am I going to take this to be the mass flow in because if you see the inside one this is rho kg per meter cube v is meter per second and area is meter square.

Incompressible Fluid $0 = \int_{CS} \rho \vec{V}.d\vec{A}$ The size of the CV is fixed

$$\int_{CS} \rho \vec{V}.d\vec{A} = \pm \left|\rho_n V_n A_n\right|$$

When uniform flow at section n is assumed

So, what you have in the one that I have circled over here is essentially kg per unit time kg per second. So, that is the mass flow rate. Now, when am I going to when am I going to use a plus sign and when am I going to use a minus sign? If you look at the definition of mass which is coming in to this system, the vectorial product tells you that the mass flow in will always be negative and mass flow out of a control volume is going to be positive. Because the area vector for a surface the area vector is always pointed outward and if you have flow this is this is if you have flow which is coming in this is the area vector and this is the velocity of the fluid that is coming in. So, if they are oppositely directed that means, the mass is entering then this expression essentially gives you that the mass flow rate in is going to be negative whereas, mass flow rate out is going to be positive.

So, that is the convention scientific notation which we will be using throughout our discussion for the next few classes in is negative out is positive. Now, similarly one can write the momentum equation for the inertial control volume. So, a control volume which is fixed in space. So, in this case n is the momentum. So, corresponding intensive property is n by m mass.

So, n is momentum means mass times velocity. So, eta the intensive property is simply going to be velocity. So, once again when the system and the control volume coincide dn dt is simply equals to the time rate of change of momentum. The time rate of change of momentum is essentially the force on the control volume. Now, these force on the control volume are divided into two to one is due to the surface example pressure other could be due to the body force example gravity.

Momentum Equation for Inertial CV, N = Momentum, η = Velocity

$$\left.\frac{\partial N}{\partial t}\right|_{SYSEM} = \frac{\partial}{\partial t}\int_{CV} \eta \rho d\cancel{V} + \int_{CS} \eta \rho \vec{V}.d\vec{A}$$

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V} \rho d\cancel{V} + \int_{CS} \vec{V} \rho \vec{V}.d\vec{A}$$

$$\vec{F}_B = \int_{CV} \vec{B} \rho d\cancel{V} \qquad \vec{F}_S = \int_{A} -p\, d\vec{A}$$

And on the left-hand side you have this where eta is replaced by v over here because of my eta in this case is equal to velocity. So, this is the integral form of the momentum equation it is identical in concept to Navier Stokes equation, but all velocities are average velocities and all velocities are measured with respect to the inertial control volume. So, this F B the body force is simply B which is body force per unit volume times g rho dv and F S the common surface force is the pressure. So, the pressure is what the pressure due pressure due to the force due to the pressure is always going to be integrated over the area minus p dA. So, once again if we do this consider this equation this is the previous equation the equation that you see on the slide right now is a vector equation.

$$F_x = F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u \rho\, d\cancel{V} + \int_{CS} u\, \rho \vec{V}.d\vec{A}$$

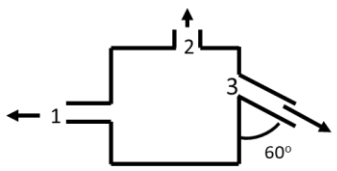
So, I can write the scalar components of this vector equation and for example, the x component of this equation is simply going to be F S x. That means, the surface force in the x direction plus F B x body force in the x direction is simply going to be del del T u rho dv essentially the velocity is replaced by the x component of the velocity in these two terms. So, u is the x component of the velocity present in the system. So, again once again to evaluate the sign of rho v dA that means, this part once again we use the same concept that the in is going to be positive and the out is going to be, sorry in is going to be negative the out is going to be positive. And then we are going to put the x the x the velocity over here the plus minus to be decided by u to find out whether it is in or out and then you are going to put the component right component.

1. To determine the sign of $\rho\, \vec{V}.\, d\vec{A} = \pm \left|\rho\ V\ \ d\,A \cos\alpha\right|$

2. To determine the sign of each velocity component

$$u\ \rho\, \vec{V}.\, d\vec{A} = u\ \left\{\pm \left|\rho\ V\ d\,A \cos\alpha\right|\right\}$$

That means, the x component of the velocity in there to obtain what is going to be the net efflux of momentum, rate of net efflux of momentum through the control surfaces into the control volume. So, this term is the net efflux of momentum, the time rate of the net efflux of momentum this is the surface force, this is the body force and this is the time rate of change of momentum inside the control volume. Once again, these forces mean they are on the control volume the forces that you see in this equation are on the control volume. So, by the control volume would simply be the negative of that. So, F that you see in this is force on the control volume.



Fluid with $\rho = 1050\ kg/m^3$ is flowing through the box, $A_1 = 0.05\ m^2$, $A_2 = 0.01\ m^2$, $A_3 = 0.06\ m^2$

$\vec{V}_1 = 4\,\hat{\imath}\ m/s$ $\vec{V}_2 = -8\,\hat{\jmath}\ m/s$

Find V_3

Now, let us try to solve quickly solve two problems to clarify whatever we have discussed this morning and then we will continue using this equation for various situations to get insights into the flow the force and so on. So, what you see is an enclosure with three inlets and outlets. So, a fluid with rho which is which is given 1050 is flowing through the box A 1, A 2, A 3 are given, v 1 is 4 i meter per second. So, even though I have put the arrow out for all these cases you can see that 4 i that means, the flow is in here v 2 is minus 8 j. So, if the flow is in here and we have to find what is v 3.

So, the arrows that you see over here are just indicators of flow moving going out or in whether it goes out or in will depend on whatever be the values of v 1 and v 2 that you would get. So, we have to find v 3 and this is simply a situation in which summation of rho i v i A i equals 0 that is the continuity equation conservation equation and since rho is constant, I am going to take rho out of this out of this equation. So, my v 1 A 1 v 2 A 2 plus v 3 A 3 would be equal to 0. We have to be just careful about what is going to be the sign of each one of them. So, v 3 A 3 is simply going to be minus v 1 A 1 minus v 2 A 2 and you put the expression for v 1.

$$\vec{V}_1\overrightarrow{A_1} + \vec{V}_2\overrightarrow{A_2} + \vec{V}_3\overrightarrow{A_3} = 0$$

$$\vec{V}_3\overrightarrow{A_3} = -\vec{V}_1\overrightarrow{A_1} - \vec{V}_2\overrightarrow{A_2}$$

$$= -4\mathrm{i}.\,0.05(-\hat{\imath}) - (-8\hat{\jmath}).\,0.01\hat{\jmath}$$

So, the v 1 is 4 I, A 1 is this multiplied by minus i because if you if this is your area A 1 the area vector is directed towards the left. So, this is the direction of the area vector and that is why I have put the minus in here. Similarly, v 2 is minus 8 j as provided in the question. So, minus 8 j and the area vector for area 2 this is my area 2 it is directed upward so that means, in the plus y direction.

So, that is why the A 2 area is positive. So, area vector always directed out and as you could see that area vector for 1 is going to be in the minus x direction. So, that is why I have the minus i and this one is going to be in the plus y direction. So, that is why it is going to be 1 plus 0 j. So, once you evaluate this the v 3 A 3 is simply going to be this much and now you could see that v 3 A 3 is greater than 0 it is a positive one. So, flow at section 3 is going to be out of the control volume and you can figure out what is the numerical value of v 3 from this.

$$\vec{V}_3\overrightarrow{A_3} = 0.28\ \mathrm{m^3/s}$$

Since, $\vec{V}_3\overrightarrow{A_3} > 0$, flow at section 3 is out of CV

$$V_3 = \frac{1}{A_3} \times \frac{0.28\ \mathrm{m^3}}{\mathrm{s}} = 4.67\ \mathrm{m/s}$$

From geometry, $\vec{V}_3 = V_3 \sin\theta\,\hat{\imath} - V_3\cos\theta\,\hat{\jmath} = 4.04\hat{\imath} - 2.34\hat{\jmath}$

So, from the geometry now you can put the complete vectorial form of the velocity v 3, v 3 sin theta v 3 cos theta and you would get the final expression for the velocity in here. So, the problem is straightforward, you figure out what is the velocity and area put the plus or minus signs accordingly and then compute what is going to be the flow rate or rather what is going to be the v 3 A 3 in this case for the unknown third control surface. So, that is everything about conservation of mass. Now, the next problem is the same thing everything remains same the same problem for which we know that v i v v 1 is equal to 4 i and v 2 is equal to minus 8 j as we have as we have calculated before. We need to find out what is the net rate of efflux of momentum through the control volume.

So, how much of momentum being added, how much the rate of momentum being added and the rate of momentum which goes out of it the algebraic sum of these two is the net rate of efflux of momentum through the through the through the control volume. So, this is the summation of inflow and outflow of momentum into the through the control volume. So, the expression as I have mentioned to you before when you think about the governing equation this efflux of momentum through the control surface is this term v rho v d A. This is essentially the mass which is which the mass which is coming in or going out and this is the velocity. So, mass times velocity this is the momentum through the control surface in or out of the control volume.

The net rate of momentum flux is given by,

$$\int_{CS} \vec{V}\,\rho\,\vec{V}\,.\,\vec{dA}$$

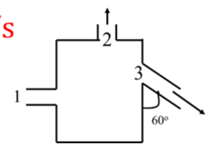
$$\vec{V_1}\rho\vec{V_1}\overrightarrow{A_1} + \vec{V_2}\rho\vec{V_2}\overrightarrow{A_2} + \vec{V_3}\rho\vec{V_3}\overrightarrow{A_3}$$

So, we will have to we will have to figure out each term three terms for surface 1, 2 and 3 and then sum them algebraically to find out what is the net rate of momentum efflux. So, once again since it is an incompressible fluid. So, therefore, summation of rho i v i A i would be 0 and the rho is a constant. So, I can I can drop the rho from here and therefore, the momentum in through the control's momentum is simply going to be the mass multiplied by the component of velocity at each one of these surfaces. And this is the mass that is going out component of the y the y component of the velocity at each of these at each of these control surfaces.

So, why this is positive negative? Why this is negative? This is negative and this is positive I have already explained to you. So, the mass which is mass is coming in through the control surface 1. So, that is why this is this is negative mass is going coming in through the control surface at 2. So, this is negative again and mass is going out of the control surface.

$$\overrightarrow{mf} = [u_1\{-|\rho V_1 A_1|\} + u_2\{-|\rho V_2 A_2|\} + u_3\{|\rho V_3 A_3|\}]\hat{\imath} +$$

$u_1 = 4\ \mathrm{m/s}$ $\quad u_2 = 0$ $\quad u_3 = 4.04\ \mathrm{m/s}$



$$[v_1\{-|\rho V_1 A_1|\} + v_2\{-|\rho V_2 A_2|\} + v_3\{|\rho V_3 A_3|\}]\hat{\jmath}$$

$v_1 = 0$ $\quad v_2 = -8\ \mathrm{m/s}$ $\quad v_3 = -2.33\ \mathrm{m/s}$

$$\overrightarrow{mf} = 349\,\hat{\imath} - 13.5\,\hat{\jmath}\ \mathrm{N}$$

So, that is why this is positive this is positive. The same applies for the for the y component of momentum being added or subtracted from the from the from the from the control volume. The one thing that we need to do right now is to put in the values of u 1, u 2, u 3 and so on. So, let us let us do that and try to try to complete the problem. The u 1 is 4 meter per second as mentioned in the problem, u 2 if you look at this point at this location there is no x component of velocity, the velocity is only in the y direction. So, that is why u 2 is going to be 0 and u 3 we have evaluated in the previous part of the problem to be equals 4.04 plus 4.04 meter per second. So, once again u 1 is provided in the problem, it comes into the control volume, it comes into the control volume that is why the mass flow rate has a minus sign associated with it, but u 1 is towards the positive x direction. So, u 1 is positive there is no velocity, no x component of velocity at 2. So, that is why u 2 is 0 and u 3 it is going out as we have calculated. So, the x component of this velocity going out of 3 is 4.04 and it is positive. Then comes this v 1, v 2, v 3 if you if you see what is happening at v 1 there is a velocity in the x direction, but there is no velocity in the y direction. So, my v 1 is simply going to be equal to 0. When you think about v 2, I have definitely a component of I mean a velocity in the y direction and that y direction velocity is minus 8 meter per second as part of the problem. The mass comes in through 2. So, if the mass comes in through 2, the velocity is going to be in the minus y direction.

So, that is why v 2 is going to be equal to minus 8 meter per second. And when I you think about the velocity at location 3, when you figure out what is the component, what is the y component of the velocity that goes out, it is also directed in the minus y direction. It is also directed towards the minus y direction. So, my v 3 is simply going to be equals to minus 2.33 meter per second. So, whatever be the velocity the y component of that in the negative y direction. So, that is why this v 3 is going to be equal to minus 2.33 meter per second. You put the all the values in there and you could calculate you would see that the effective momentum the net momentum efflux into the control volume due to the flow in and out of 1, 2 and 3 turns out to be about 350 Newton.

So, 350 I minus 13.5 J Newton. So, that essentially tells you what is the net addition of momentum to the control volume because of flow of momentum through the control surfaces 1, 2 and 3. So, this essentially explains this comes to we come to the end of this lecture and we understand what are the forms of the basic equations in integral, what are the integral equations, the conservation of mass and the conservation of momentum. Why the mass in is going to be negative, mass out is going to be positive. And one more very important part which we you will see in subsequent lectures is that all the forces that you calculate using these integral equations are forces on the control volume.

So, that is going to be an important lesson for us. So, that is the summary of what we have covered today. Thank you.