Momentum Transfer in Fluids
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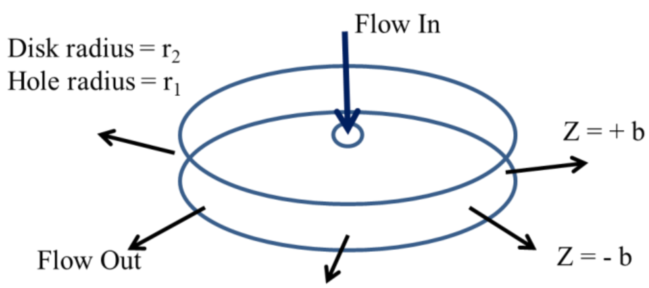
This is going to be the last class on the topic Navier Stokes Equation. So, far we have seen the choice of the component, right component of Navier Stokes equation is the first step in solving any momentum transfer problem. But there is another class of situations in which it is not just the momentum equation or Navier Stokes equation which is to be considered. Sometimes we need to start with the equation of continuity as well because by nature the equation of continuity can give us some idea, approximate idea about the dependence of the velocity on more than one independent variables. So, this kind of situation arises when the velocity is function, is a function of let us say both x and y. So, when there is multidimensional dependence of velocity then it is better to start with equation of continuity first and then get an approximate idea of the functional dependence of the one component of the velocity and proceed with the Navier Stokes equation by choosing the right component in such cases.

So, far we are we also have dealt only with situations in which the flow area remains a constant, the flow area did not change. But if you think of a multitude of situations in which the flow area could be a function of position as well. So, we are going to deal with the last problem of this topic on a situation where the flow area keeps on changing in the direction of flow as well as the right choice of Navier Stokes equation must be preceded by the first working with the equation of continuity. So, these two are the major features of today's problem, the last problem that we are going to discuss in Navier Stokes equation or equation of motion.

So, from next class we will start a completely new topic, but this I kept this problem for the last class for on this topic. So, that you would be you are well conversant on the use of Navier Stokes equation by now, but this is something different. So, the problem that we have here if you look at this problem it is two plates one on top of the other. So, we have two cylindrical plates or one on top of the other with a certain gap as you can see in the figure it is the gap between the two figures is equal to twice B. So, z equals plus B and z equals minus B and the disk radius is going to be equal to r2.



The pressure is p_1 at $r = r_1$ and p_2 at $r = r_2$

So, the point here is situated the point here is situated at a distance r2 from the centre, but the main difference in this problem is there is a hole on the top cylindrical plate right at the centre with a radius equal to r 1. So, the radius of this hole is equal to r 1 whereas, the radius of the disk of any of the disks is equal to r 2. Lubricant at high pressure, this mostly it is going to be a lubricant at high pressure flows in the space in between the two disks and once it comes through the hole at the top then it starts distributing radially outwards in all directions, in all possible directions. So, the flow comes in and then starts going out through all possible directions in the r direction. So, you can figure out that the one of the principal components that we have to consider here is v r.

So, v r is going to be the principal component of velocity which we have to which we have to evaluate, but one can very legitimately ask the question is what is going to happen just below this hole, would not there be a v z component, could there be a v theta component as well because the flow is coming from the top through the hole and once it crosses the hole then it starts spreading in all, in the r direction. But what is going to happen right at this point where the flow is going to come from the top and it hits the bottom plate. So, there is going to be multidimensional effects right over here with uncertainties of which component is going to predominate, is it going to be v z, is it going to be v r, can there be some swirling component of velocity. So, can there be some v theta as well. So, this uncertainty part, this part of the flow where it is quickly, where it is developing and not completely developed, we keep outside of our analysis.

What we can say is that all the disturbances that I have described so far, all the disturbances that are bound to happen is going to happen within the region 0 to r 1. So, the flow quickly stabilizes in this region between 0 to r 1. Once it crosses r 1 then it is going to flow in the radial direction in a very orderly laminar fashion. However, it is important that we identify the limitation of our analysis is that right below the hole where the lubricant where the lubricant is entering at a high pressure, the flow may not be laminar, there can be intermixing, there can be other components of velocity which will be significant and comparable in magnitude to v r, the r component of velocity. And therefore, our simplified analysis is not valid for any region whose radius is less than r 1.

Now, I said that the situation is most likely to happen, the geometry that you see is most likely to happen for the case of two bearings which are separated by certain distance and you apply lubricant at high pressure and let that leak such that the two plates do not come in contact and have create friction. In many situations one of the plates is not going to be stationary, it will rotate at a high speed. So, it is the flow of the lubricant the presence and the leakage flow rate of the lubricant which keeps the two plates separate reduces the frictional wear and tear. So, it is extremely important for us to know, for us to suggest what is going to be the flow rate, the leakage flow rate and what are the component, what are the parameters be it geometric, be it operational or the properties of the lubricant on which this velocity distribution or the leakage flow rate depends on. So, that is the genesis of the problem that is the reason why we are interested in obtaining a solution for this specific case.

So, with that background now let us start trying to figure out what is going to be the velocity distribution in the space, in between two disks where of course, there is going to be a huge pressure gradient that forces the fluid to move in the r direction. There is not going to be any body force because the disks are horizontal, but a pressure gradient forces the liquid from the inside to the outside, to the periphery of the two disks and what is going to happen to this pressure difference. Is it going to be linear or is it going to be is it going to depend only on r or can that depend on something else as well. So, those are the questions which our analysis should answer. So, we start the solution, but this time since we are unsure about the about the dependence, is the v r going to be a function of z or it could be a function of r as well.

So, when you think mentally think about the space in between two cylinders, the velocity is going to be a function of where it is located in terms of the z direction. But as you move outward the annular area which becomes available for flow keeps on increasing because the flow area at any value of r is simply going to be twice pi r times delta H where delta H is the separation between the two plates. So, it is a ring like area whose one of the dimensions is going to be twice pi r, the other dimension is going to be the separation between the two plates which according to the geometry of that we have for this problem is going to be equal to twice B. So, the area is area for flow is twice pi r times twice B. Now, note that this r is small r and this r keeps on changing between values equal r 1 to r 2.

So, in between r 1 and r 2 the flow area for the lubricant keeps on increasing and we know that as area increases the velocity has to decrease since it is incompressible fluid. So, rho A 1 v 1 is to be constant that is what conservation of mass is. So, if I take out the rho since it is constant, A 1 v 1 must be a constant. Now, in all our previous problems the area available for flow was constant, it was not changing. So, v 1 so, in, but in this specific case what we have is that my area the flow area keeps on increasing.

So, the velocity has to decrease. So, v r will decrease as we move to larger and larger values of r. So, v r addition in addition to being a function of y, that means, it is going to be 0 at z equals minus v in order apart from being a function of z according to the figure that we have. So, v r is going to be a function of z at z equals minus v and z equals plus v the velocity due to no slip condition will be 0. However, this v r is also going to be a function of r.

So, this is the first case of multidimensional dependence of velocity that we are encountering. So, when we think about the problem analyse the problem mentally before getting on to the solution, we believe that it is a multidimensional problem, v r is a function of z and v r is a function of r. So, if such is the case then the first point to start would be equation of continuity. So, we start with equation of continuity and see if that equation can provide some information to us which will subsequently be useful in solving the Navier Stokes equation. So, that is the reason why we start with equation of continuity in this case and not directly with equation of motion.

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)=0 \qquad \rightarrow \quad r v_r = Constant; \quad v_r=\frac{\phi}{r}$$

So, if we identified that it is the v v r is a function both of z and r. So, we start with the equation of continuity. So, the equation of continuity essentially for a cylindrical coordinate system is this, where the first term refers to the steady or unsteady nature of this situation and we know that this is a steady state case. So, this equation this term is simply going to be equal to 0. Next is that there is no v theta component, the there is no flow in the theta direction the flow is only in the in the in the r direction.

So, this part will also be 0 and finally, there is no v z component even though v r depends on z v r depends on r, but there is no v z component except near the hole through which the flow is entering the space in between the two disks, but we are not considering that part at all. So, from 0 to r 1 is the region which we do not consider where there can be other the v v there can be a non-zero v z as well. One may ask is why are you saying that beyond r why would everything die down at r equals r 1, there could be this unstable region can extend beyond r equals r 1 as well, very true it may be possible that the zone of disturbance due to the introduction of fluid from the top will not be confined in the region defined by the whole dimension that r is equals r 1. So, that is an approximation which you are using here, but our approximation is justified to some extent since we are dealing with a lubricant and high viscosity. As I have mentioned before, viscosity brings order back to the system.

So, even though there are disturbances the viscosity high viscosity does not allow these disturbances to amplify these disturbances to exit. So, it dampens all these disturbances and therefore, you would get you would get a situation where the flow is again going to be one dimensional only. So, with that now we go into this we go into this solution of this. So, only one term in the equation is therefore, going to be relevant. So, with that we set this to be equal to 0 and integrating it once we get r v r to be a constant and let us call that constant as phi.

So, phi is the constant note here that I was not able to drop the partial sign, I have to use del del r. Why did I have to use this? Because I understand that my v r is a function can be a function both of r and of z. So, that is the reason why I had to keep the partial differential sign in there and what we get from my equation of continuity is v r to be equals phi by r. Now, there is something interesting about phi, phi is an integration constant. So, but if you look carefully this phi has been obtained when you started working with a partial differential equation and obtained phi as the integration constant.

We know that v r is a function of r and z. So, v r is a function of r and z and in this equation del del r of phi del del r of rho v r is 0. So, if that is the case then it tells me that phi has to be a function of z. So, the reason that we have started with working with the Navier Stokes equation is that it gives me a form of the expression of velocity v r is going to be phi by r. From our analysis we know that v r is a function of z and r and since del del r, since del del r of r v r del del r of r v r is equal to 0 that means, del phi by del r is equal to 0 which gives me phi is a constant and phi is a function of z only.

So, equation of continuity gives me the approximate form for the expression for velocity where we understand that the integration constant of the partial differential equation is a function has to be a function of z. So, that is the important information that we have obtained from the equation of continuity. So, now, we are in a position to start with Navier Stokes equation and of course, the flow is in the r direction. So, I have to choose the Navier Stokes equation in the r direction which looks like this. This you have this we have discussed many a times.

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right) \qquad \frac{\partial}{\partial r}\left(r\,v_r\right)=0$$

$$=-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]+\rho g_r$$

So, this is the r component of the Navier Stokes equation. The first one would simply be equal to 0 since it is steady state. So, that part is going to be 0. I cannot neglect the second term because v r unlike all the problems that we have discussed before, v r is a function of r. So, the second term has to remain in the Navier Stokes equation that is one big difference from the previous problems.

What happens to the next one? My v theta is 0 there is no component of velocity in the theta direction, no component of velocity in theta direction once again and even though v r is a strong function of z the distance between the two plates that means, v r is a strong function of z, but v z is going to be equal to 0. So, of the left-hand side convective transport of momentum one term will be nonzero. Therein lies the difference with the previous problems. Let us come to the right-hand side. What I have on the right-hand side? The first term is imposed pressure gradient.

Unless there is a strong pressure gradient imposed in this system the high viscosity lubricant will never flow. So, therefore, the there has to be a pressure gradient and that is why this d p d z is not going to be d p d z is not going to be 0. The second one is v theta will not be a function of theta and therefore, this del v theta and v theta does not exist. So, this del v theta del theta will be equal to 0. Now, del square v r del theta square because of angular symmetry this term will also not be present.

So, that leaves us with these two terms in here. One is del del r of r v r and the other one is del square v r by del z square and we understand that v r is a function of r and z. So, I am unsure at this point how do I deal with these two terms inside the third bracket, but if I concentrate and there is obviously, no gravity. So, that rho g r part would disappear, but if I constant within the bracketed part del del r of r v r what does that tell us? Del del r of r v r can I assign some value to it? Go back to the equation of equation of continuity once again and you could see you would see that del del r of r v r is equal to 0. So, again this underscores the importance of equation of continuity in simplifying the Navier Stokes equation and in certain cases.

So, this r v r del del r of r v r is equal to 0 which comes directly from equation of continuity. So, my complicated looking Navier Stokes equation for a not so simple case turns out that I have only three terms to think about in the equation of in the Navier Stokes equation, the first second and the third term. So, if I look at my governing equation now the governing equation would be would look like the first term on the left-hand side which comes this term on the left-hand side where the contribution is from convective transport of momentum. The second term which is the applied pressure gradient and the third term is the viscous transport of momentum in the system. So, these are the three different terms which one has to consider and we also understand from my equation of continuity that the form of v r is going to be phi which is a function of z that we have discussed divided by r.

$$\rho\ v_r \frac{\partial v_r}{\partial r} = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2} \quad ; \quad v_r = \frac{\phi\ (z)}{r}$$

$$-\rho\ \frac{\phi^2}{r^3} = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2}$$

So, when I substitute this expression for v r into my equation of into my Navier Stokes equation this is the final form of the governing equation that dictates or that describes flow in the z flow in the flow in the r direction for two discs very close to each other this is going to be the governing equation. The once again in order to obtain in order to solve this it is not possible at this moment this is a partial differential equation and it is non-linear also the non-linearity in this equation comes from the left-hand side comes from this term which is due to convective momentum. So, we have to make certain approximations in order to obtain a closed form analytic solution for this and to justify it. Once again now I go back to the to my description of the description of the problem where I said that this situation is going to be more common where we have a flow of a lubricant in between two plates with the sole purpose of having a thin film of lubricant in between the two plates one of them could be moving. So, if this is a lubricant that is flowing in between the two plates, we would like to use the least amount of lubricant.

So, we will apply the ideally what is going to happen is we are going to have a thin film of lubricant which flows with a very small flow rate in between the two disks. So, my ideal condition is a very slow flow of the lubricant as a result of applied pressure gradient and the lubricant is highly viscous. Now, when the flow is very slow, but the gap is extremely small, but when the flow is slow that means, which component of momentum which type of momentum transport is going to be going to be weak and which is going to predominate. Of course, convection which is associated with the velocity of flow with the movement actual movement of the fluid the convective contribution to the overall momentum transfer is going to be minimal as compared to the viscous transport to momentum transfer. So, if you think about v r, v r itself is small, but the delta z, the gap between the two disks is extremely small.

So, the del square v r by del z square can be substantial. So, that is something which we have to keep in mind and even the phi that means, the velocity is small that means, phi is small, but r is not small. So, phi square by r cube is small as compared to del square v r by del z square. So, that is the order of magnitude analysis of the equation of motion which you can do based on the specific application that you are dealing with. So, this physical reason for this is the convective contribution to momentum is small as compared to the diffusive transport of momentum which I have written over here.

In the idea in the imaginary case when the convective effects are very very small then the entire LHS of the equation can be set to 0. And when you make that assumption that the convection convective transport of momentum is not present at all or his contribution is very very small that type of flow is known as creeping flow, where the flow is creeps where the flow velocity is extremely small. So, there is no convective transport contribution to the Navier Stokes equation and that idealized condition when it approaches 0 is known as creeping flow, which as I mentioned why it could be a justifiable approximation for flow in between two disks of a high viscosity lubricant. So, with this then my governing equation simply becomes d p d r and we also understand that since the gap is very small p is not going to be a strong function of z, p is going to be a function only of r. And my partial differential equation has now been converted to an ordinary differential equation, where it is simply going to be d 2 phi d z square by and this is going to be the functional form the final form of this equation.

$$0=-\frac{dP}{dr}+\frac{\mu}{r}\frac{d^2\phi}{dz^2} \quad ;P \text{ is a } fn \text{ of } r \text{ only}, \phi=f(z) \text{ only}$$

So, with this equation let us assume that we have a constant applied pressure difference. So, my equation turns out to be d p d r is this, p is a function only of r and not of z. So, we can figure out what is going to be the delta p and we understand that at r equals r 1 p is p 1 at r equals r 2 p is equal to p 2. So, upon integration we get a pressure distribution in the flow field, note this distribution in here carefully. So, for the first time what you are seeing is that the pressure difference is not linear, but it is logarithmic in nature.

$$\frac{\mu}{r}\frac{d^2\phi}{dz^2} = \frac{dp}{dr} \qquad \Delta p = p_1 - p_2$$

$$\mu\frac{d^2\phi}{dz^2}\int_{r_1}^{r_2}\frac{dr}{r} = \int_{p_1}^{p_2} dp$$

$$\mu \ln\frac{r_2}{r_1}\frac{d^2\phi}{dz^2} + \Delta p = 0$$

And this happens since the flow area keeps on increasing flow area keeps on changing in the direction of flow. Had it been remained constant this d p d x would be a constant, but since it is not the analysis tells you that it is going to be logarithmic in nature. So, with this you can evaluate what is phi and the phi is and the you can use the boundary conditions, the no slip boundary conditions at plus B and minus B the velocity is going to be equal to 0. So, this is the no slip condition. So, with this you can evaluate C 1 and C 2 and we understand that v r is simply phi by r as we have shown before.

$$\phi = -\frac{\Delta p z^2}{2\mu \ln\frac{r_2}{r_1}} + C_1 z + C_2$$

Boundary Conditions:

$$\left.\begin{aligned} &\text{At, } z = +b \text{ and} \\ &z = -b \end{aligned}\right\} v_r = 0$$

$$v_r = \frac{\phi}{r} \qquad v_r(r,z) = \frac{\Delta P b^2}{2\mu r \ln\frac{r_2}{r_1}}\left[1 - \left(\frac{z}{b}\right)^2\right]$$

So, this gives you the complete velocity profile for flow between two discs when we can assume that it is a creeping flow situation. So, this is conceptually an involved one and I try to explain it to you the steps to you as clearly as possible. This problem is solved in Bird-Stuart and Lightfoot. You can look at the complete solution and once again I argue that you do this do this analysis on your own get this form of the velocity distribution and in the book of bird-Stuart Lightfoot you will get a step by step at least some of the steps in order to arrive at this expression. So, finally, this is my velocity and once I have the velocity then if I have to find out the leakage flow rate, I am simply integrating this v r over the over the area.

$$v_r(r,z) = \frac{\Delta P\, b^2}{2\,\mu\, r \ln \frac{r_2}{r_1}} \left[1 - \left(\frac{z}{b} \right)^2 \right]$$

$$Q = 2\pi \int_{-b}^{+b} r\, v_r\, dz = 2\pi \int_{-b}^{+b} \phi(z)\, dz = \frac{4\pi\, \Delta P\, b^3}{3\,\mu \ln \frac{r_2}{r_1}}$$

So, the area available for flow is twice pi r del d z and in v r, I put this expression for v r, I integrate it between plus B to minus B and what I get is this expression for q and as you could see intuitively my expression is correct as I increase my delta p my q should increase. If I increase the viscosity if it is more viscous then q should decrease it is more difficult to make the fluid flow. If the gap increases, then there is going to be more flow. So, whenever you figure out any expression try to mentally think if it conforms to your understanding that if you vary the if you vary one of the parameters where whether or not according to the expression that you have derived whether it behaves as it should. So, this concludes our discussion on the Navier-Stokes equation and its applications.

So, in that from the next class onwards we will start a different topic, but there would be assignments at every step and I again request you to solve the assignments and if there are any difficulties you can always come to the forum and put the questions for the T A and me to respond to. Thank you very much.