Momentum Transfer in Fluids
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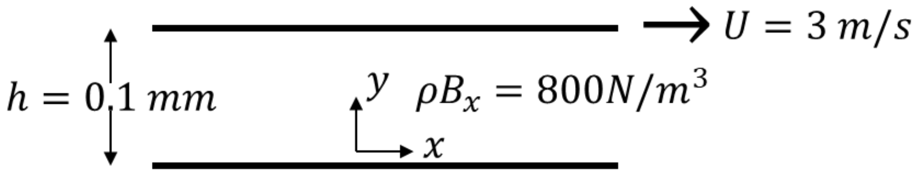
Good morning. This is the penultimate lecture on the topic Navier-Stokes equation and its application. So, in this class we are going to see two problems of two different types and the final class on the application of Navier-Stokes equation would be the next one in which we would see a situation in which the velocity could be a function of two dimensions. So, but for this one we will see a problem first in which the body force is not gravity, but something else. So, it is just a question of replacing the body force, replacing the gravity with some other kind of a body force which is electrical in nature. So, let us first go to the statement of the problem.

What we have is flow between two parallel plates as before where the top plate is moving with a velocity as you can see of 3 meter per second, but it is a horizontal system. So, normally we do not associate anybody force with a horizontal system since mostly we deal with gravity as our sole body force. But in this specific case there is a body force rho B x 800 Newton per meter cube which is acting in the direction of flow. So, this body force and the motion of the top plate would sustain a flow in the positive x direction.



So, what we need to figure out is what is the volume flow rate past a vertical section. So, if we draw a vertical section what is going to be the flow rate past this section. One point to note another two points that we should be would should be careful about. The first thing is as I mentioned previously all terms in Navier Stokes equation are force per unit volume. So, whenever you are including a new force into the system, into your governing equation make sure that it is in Newton, I mean force per unit volume.

So, what we have here is this rho B x is given in Newton per meter cube. So, we can directly add this force into our Navier Stokes equation in place of the gravity force. So, that is the only difference from all the problems of Couette flow associated with gravity etcetera. And secondly, the application of this kind of situation where would you get this kind of situation. So, this application of an electric field would cause a mobility, electrophoretic mobility into the ions which are present in the solution.

So, because of this because of this applied electro electric field the ions are going to move in a specific direction depending on their charges. When the ions move, they will also drag liquids along with it. So, that would create a motion in the entire liquid and a flow is going to generate and this flow though small can be very useful in systems which are very small in size. So, for some microfluidic applications this kind of application of an electric field can create a flow even without the presence of presence of an applied pressure gradient. So, we would deal with this situation, but in order to do that this is the schematic of the problem that we are going to deal with and it is essentially Couette flow in presence of a body force.

$v_y = v_z = 0$ $\qquad\qquad$ $v_x \neq f(x)$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x$$

$0, SS$ $\quad$ 0, continuity eqn $\quad$ No imposed pr. gradient $\quad$ $v_x \neq f(z)$

So, we are going to first start writing the equation Navier Stokes equation the proper component of the Navier Stokes equation and cancel the terms. So, here the flow is in the x direction. So, I am going to write the x component of Navier Stokes equation as you can see in your slide and I have and I have crossed out certain terms with a very small explanation for each one of them. So, for the first term you have steady state situation. So, therefore, the velocity does not vary with time.

The second term the del v x del x equals 0 it comes from continuity equation because if you recall the continuity equation is del v x del x plus del v y del y plus del v z del z would be equal to 0. Here there is no v x no v z. So, del v x del x must be 0. I write the equation just to make it complete del v x del x the equation of continuity for an incompressible situation plus del v x sorry del v this is equal to 0 there is no v y no v z. So, del v x del x would be equal to 0.

So, continuity equation tells us that the velocity in the x direction it does not change. So, if you specify the y location if you specify any y location the velocity is going to be velocity in the x direction is going to be the same across all these points as long as the y is fixed. So, therefore, del v x del x is 0, v y and v z both are 0 there is no imposed pressure gradient in this system. So, it is just a simple flow problem v x is not a function of x. So, that makes this term to be equal to 0 and then you also have v x is not equal not a function of z.

So, this term would be 0, but instead of rho g x which is the gravity for the body force the gravity the rho g x is replaced by rho b x which is the electrical force which is the electrostatic force that is also a volume that is also a body force acts on the entire volume every point on the fluid. So, my governing equation the governing equation would simply become this and the governing equation would be just this term and this term that is going to be equal to 0 and the what are the boundary conditions going to be the boundary conditions are going to be the no slip conditions that is at y equals 0. That means, on this plate the v x value of v x is going to be equal to 0 since it is a static plate and at the liquid solid interface the relative velocity is 0 which is the no slip condition. And similarly at y equals h that means, at the top plate, the velocity must be equal to the velocity of the plate which is moving. So, the no slip conditions at y equals 0 and y equals h would allow you to find out what is the velocity distribution.

$$v_x = \frac{Uy}{h} + \frac{\rho B_x h^2}{2\mu}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]$$

And once again this is purely qth flow term if only the top plate was moving then the velocity profile would be linear and the velocity expression the variation of velocity with y would be would look like this. The second term is due to the rho b x term the body force due to electrostatic it is an electrostatic force and this is going to be the distribution of total distribution of velocity one for the qth flow and the second term for the body force. Once you have the velocity the point velocity then in order to obtain the flow rate you need to figure out first what is going to be the average value of the velocity. And if you if I do this problem for per unit width basis so, my unit width is going to be simply equals to 1 and where d z is the depth and it varies from 0 to 1 and, but my v x is going to vary from 0 to h. So, if you look at dy d z that is essentially the flow area d A.

$$\langle v_x \rangle = \frac{1}{1 \times h} \int_0^1 \int_0^h v_x dydz \ \rightarrow \langle v_x \rangle = \frac{U}{2} - \frac{\rho B_x h^2}{12\mu}$$

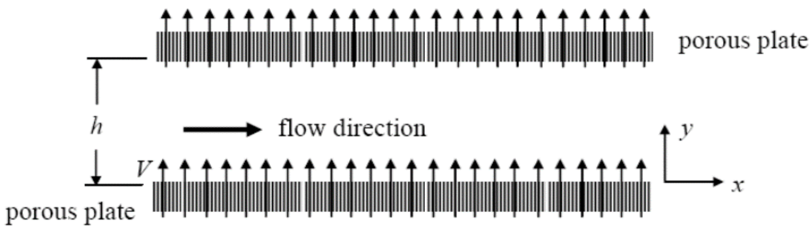
$$\text{Volumetric flow rate, } Q = \langle v_x \rangle \times h \times 1 = 1.5 \times 10^{-4} m^3/s$$

So, it is an area averaged velocity where I have taken the d z to be of unit depth. So, if you do that than your average velocity is going to be this and the average velocity can be can be converted to volumetric flow rate simply by multiplying it with the area and once again because of the unit depth the value of the value of z is simply or width is simply going to be equal to 1. So, this is going to be the total flow rate per unit width for Couette flow superimposed on an electric field driven flow. So, that is an example where I can have a different body force in the system. So, the next problem is slightly different where apparently it would seem that I have two velocity components.

In fact, I have two velocity components. So, let us first think about the situation I will explain where you would expect this kind of situation in practice and then we move on. Now, many of you are aware of the membrane separation process where you there is a porous membrane semi permeable membrane and you pass a solution containing some solute particles dissolved in it. The size of the solute particles is larger than the size of the solvent particles. When they pass through the porous membrane because of an applied pressure the solvent comes out of the pores of the membrane, but the solute is retained.

So, the solute if you think of the solution which is passing through the solutes cannot pass through whereas, the solvent comes out of the system. So, therefore, at the end you have a solution which is concentrated which where the concentration of solute has increased. So, this kind of membrane separation process are used in many fractionations process were based on the size you can separate larger sized particles from smaller sized particles. You can use to concentrate something for example, if you would like to concentrate fruit juice you are going to force make the pulp flow through a semi permeable membrane where the solvent is going to come out while leaving the solute particles in there. So, the problem that we are going to deal with is something which you would may see in the membrane separation or similar such processes where the solutes are retained and the solvent is passing through.

Now, when the solutes are retained for quite some time then there is going to be a layer of rejected solute particles on the membrane surface which are going to clog the pores of the membrane. So, in order to wash away the deposited solute particles which are reducing the permeability the ability of the solvent to flow through the pores you apply the force apply the solvent in the reverse direction such that the pores are going to get unclogged and the solute will be driven away from the pores and therefore, you can reuse it. Consider this situation as depicted as depicted in this figure. So, what you have is a porous plate and you have a flow of a fluid in this direction, but at the same time you also have a flow in the vertical direction flow in the vertical direction through the pores over here. Now, the top plates are also having same sized pores, same size distribution and distribution of pores.

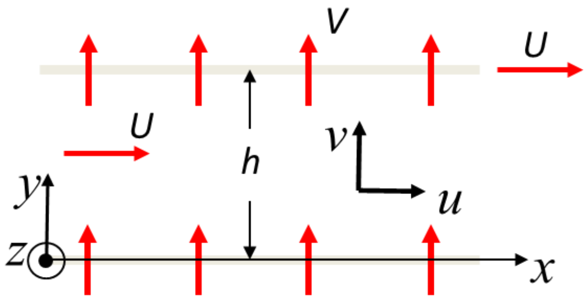


So, the fluid is going to come in the y direction as you could see here and then they are going to pass through the pores at the top while the principal flow is going to be in the x direction. So, with this now let us try to see how what is this statement of the problem. So, you have an incompressible fluid which flows between two parallel porous flat plates as you can see in the figure an identical fluid. So, if it is water which is flowing in the x direction the same solvent that is water is being pumped through the pores from the bottom leaving at the top. We need to assume that the flow is steady it is fully developed.

So, the velocity in the x direction velocity in the x direction is not going to be a function of x as I have explained to you what fully developed flow means. So, it is the pressure gradient in the x direction is a constant and it is a horizontal system. So, we can neglect the body forces. So, the pressure gradient is dp dx is a constant and there is no body force. We need to find out the expression for the y and the x component of velocity.

So, the velocity initially it is injected is capital V, but we have to figure out how does v varies with y and the velocity in the x direction how does it vary with the variable. For example, in this case it is going to be how does vx vary with y. So, we start this problem at steady fully developed flow and as a result of which nothing changes with time everything the velocity etcetera will be fixed with respect to time and the channel is quite wide. So, we do not have to consider the z direction flow in the z direction at all. So, we could assume that the flow is not dependent on the z direction which is written over here.

So, in the x direction the flow velocity u it is definitely a function of y. In the y direction there is a velocity v and v could be a function of y which we would like to ascertain, but obviously, it is not going to be a function of x and in the z direction there is no velocity and none of the component's u or v depend on the z direction. So, z direction has no role to play in this specific problem it is the x and the y direction and the variation of u with y and the variation of v with y again these two dependents' dependences we have to evaluate. So, what do we do this is once again the pictorial representation x and y and this velocity u the height y and the cross velocity is going is taken to be equal to v and what you have here is the top plate through which the fluid is the fluid is again passing through leaking through. So, we need to use first the equation of continuity in Cartesian coordinate system as I wrote to you before as I showed you before del u del x del v del y etcetera would be 0.



$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \Rightarrow \frac{\partial v}{\partial y}=0$$

So, here because of fully developed condition del u del x is going to be equal to 0. Since the flow is fully developed u is not a function of x. So, this term would disappear there is no there is no flow there is no velocity in the z direction. So, w the z component of the velocity is 0 which leaves us with del v del y to be equal to 0 which essentially gives me v is a constant and or v could be 0. Now, we know that v is equal to capital V small v is equal to capital V at y equals 0.

So, over here the velocity in the y direction is equal to capital V. So, therefore, it is going to be v everywhere in the flow field since del v is not a function of y. Since v is not a function of y. So, therefore, small v the velocity in the y direction is going to be a constant and equal to the injection velocity v with which the flow the fluid is pumped in into the system with a velocity equal to v in the in the in the y direction. So, that is a that is a first understanding and here in you see that sometimes we need to use the equation of continuity.

So, far in our previous problems we started with the Navier Stokes equation, cancel the terms and kept on solving it until we get the equation of the governing equation and then solve the equation with the solve the equation with the appropriate boundary conditions. But here is the first example where you would see that we need sometimes at times we need to use the equation of continuity as well which can give us important information about the dependence or about the nature of the flow. And in the second class in the last class on this topic I would show you how in sometimes you need to start with the equation of continuity so as to get some idea of some approximate idea of the functional dependence of one of the velocity components with other system parameters. But here in what we see is that small v is equal to capital V. So, the y component of the velocity is going to be same everywhere in the flow field.

$$\left.\begin{array}{ll} x \text{ direction:} & u = \textbf{function of } (y) \\ y \text{ direction:} & v = V \\ z \text{ direction:} & w = 0 \end{array}\right\} \quad (2)$$

So, if we express it in a compact form what we understand then is u is a function of y that we that we still have to evaluate. However, we have evaluated the y component of velocity and the y component of velocity is simply going to be equal to u equal v equals capital V and this will remain unchanged that there is no variation of the velocity with respect to the z direction. So, next comes the next comes this equation that we first need to write. If you look at this problem once again it is the x component the principal flow is in the x component. So, it is the x component of equation of motion or Navier Stokes equation that we need to write.

So, we write the x component of the Navier Stokes equation and you see the first term the first term over here is 0 since it is a steady state situation. The second term is going to be 0 since it is your equation of continuity tells you that for a fully developed flow u would not be a function of z. I cannot say anything about that I need to say something about the third term I mean third term, but I will come back to it later, but this one u is not going to be a function of function of z. Then this u is not a function of x u is not a function of z and there is no body force present in this system. So, my initial analysis of Navier Stokes equation tells me that these are the terms which I can safely neglect.

x - component: v = V

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}+u\cancel{\frac{\partial u}{\partial x}}+v\frac{\partial u}{\partial y}+w\cancel{\frac{\partial u}{\partial z}}\right)=-\frac{\partial p}{\partial x}+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}+\cancel{\frac{\partial^2 u}{\partial z^2}}\right)+\cancel{\rho g_x}$$

y - component:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

So, but what there are still few terms left in this equation. So, how do we deal with it, but once again the y component is not going to be relevant in this case since, I already have evaluated how does v change changes with y. So, my velocity distribution which is essentially velocity in the y direction v y is equal to v. So, this dependence I already have obtained from my equation of continuity. So, I do not need to work further with the x component of the Navier Stokes equation.

It is the sorry y component of the Navier Stokes equation. It is the x component which is going to give me my second objective that is to obtain u as a function of y. So, what I do next, I identify that my small v the velocity in the x direction is a constant which we have obtained over here before from my equation of continuity. So, my small v is going to be equal to capital V. So, that is the additional information that my equation of continuity has provided me.

$$\rho V\frac{du}{dy}=-\frac{dP}{dx}+\mu\frac{d^2u}{dy^2}$$

So, now, I am in a position to see what is going to be my governing equation. So, my governing equation is simply going to be rho v this term rho instead of small v I write capital V and then for del p del x which has been which has been stated in the problem to be a constant. So, it is going to be simply dp dx where p is a function only of x and not of not of y. This assumption is correct in many situations because if you if you look at the drawing if you look at the geometry of the conduit through which the flow is taking place, x is very large as compared to y. So, the only variation in the only variation of pressure in the y direction is caused by gravity the hydrostatic pressure difference, but because of the small value of y, because of the small value of the gap del p del y is usually very small as compared to del p del x that is the del p del x is the pressure different you have to impose in order to in order the for the flow to take place.

So, dp dx del p del x is quite large as compared to del p del y. So, in most of the situations what we have encountered so far, del p del x is substituted by dp dx and it is a constant and pressure varies linearly with x and that is why dp dx is a constant. What we would see later is that that the pressure remains a linear function of x that is dp dx is a constant as long as the flow area does not change. So, if you have a flow between in a tube with constant diameter a flow between two parallel plates with constant separation the pressure is not going to be a pressure is going to be a linear function of x. However, if the cross-sectional area keeps on changing then the pressure may not the pressure variation may not remain linear, but we will come to that.

Boundary Conditions
At y = 0, u = 0
At y = h, u = 0

$$u_x = \frac{h}{\rho V}\left[\frac{\partial p}{\partial x}\right]\left[\left\{\frac{\left(1-\exp\left(\frac{\rho V y}{\mu}\right)\right)}{\left(1-\exp\left(\frac{\rho V h}{\mu}\right)\right)}\right\} - \frac{y}{h}\right]$$

So, suffice to say for this specific problem pressure is a function of x only and that and dp dx is a constant, but do not take it as if the pressure is always the pressure gradient is always going to be a constant that may not be the case it is it will it will definitely not be the case when the pressure changes when the flow area changes and therefore, dp dx may not be a constant. Now, the this is the governing equation and in the governing equation we know that the boundary conditions tell me for the x component of velocity at y equals 0 the velocity is going to be 0 and y equals h the velocity is going to be velocity is going to be again equal to 0, but this brings us to one special situation of no slip condition. Here we have the situation in which the there are holes in this over which the small v which is equal to capital V is flowing and it is leaving and here you have u which is a function of y only. So, when we think about no slip condition for u no slip condition for the x component x component of velocity the velocity here is going to be 0 the velocity here is going to be 0. So, even though I have a nonzero v component nonzero y component of velocity at y equals 0 that means, over here and at y equals h my no slip condition for u will still be that at y equals 0 and at y equals h the velocities are going to be 0.

So, no slip conditions are valid for each component of it each component of the velocity. Here I do see that the condition for the y component of velocity tells me that at y equals 0 v is not 0, but that does not mean at the same point u cannot be 0. So, the no slip conditions are to be separately seen for each component of the velocity and their proper values are to be ascertained in these kinds of situations. So, what I have then is this these are the two boundary conditions and once you solve this once you solve this you would get this is the final solution for the velocity distribution that you are going to get and I once again I advise that you use you do this problem you do this problem as an exercise on your own. The complexity in the problem arises from nonzero value of v on the left-hand side.

So, this is the first example that we are dealing with in which the convective part of the Navier Stokes equation that means, the left-hand side of the Navier Stokes equation has a role to play. So, this so, for all the problems that we have encountered the entire left-hand side was equal to 0, but this is the first case in which because of the nonzero value of small v the y component of velocity there remains a term representing convection from the Navier Stokes equation from the left-hand side of the Navier Stokes equation. So, this makes the governing equation slightly different from what we have done before, but it is still situation in which a closed form solution for u x can be obtained and it is a very simple integration with the appropriate boundary conditions you should be able to solve this and for your reference I have given you the final form of your final form of what u x going to be. So, here is an example a practical example where you have cross flow present in the system and you have suction at one suction at this point and injection at this point injection at bottom and suction at the top which affects the x component the velocity distribution in the x direction. So, a nice example of contribution of convective momentum and diffusive momentum which is this term along with the presence of a pressure difference and once again the pressure difference the gradient is a constant as long as the flow area is constant.

So, these factors are to be kept in mind the convection the diffusion the pressure no body force and the governing equation with no slip condition no slip for one component of the velocity equally applicable for the other component of velocity. So, v y is not 0 at y equals 0, but v x is 0 at y equals 0. So, this distinction is to be kept in mind. So, that is all for today's class and the last class on this topic Navier Stokes equation will be the next one. Thank you.