Momentum Transfer in Fluids

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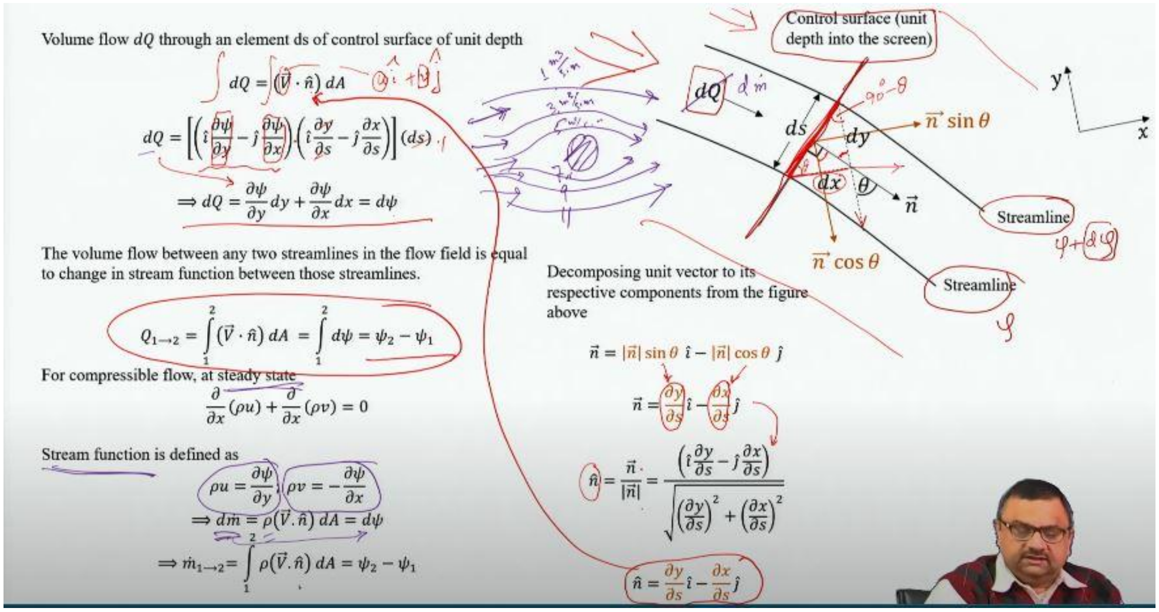
Week-05

Lecture-25

I welcome you to this lecture on Momentum Transfer in Fluids. What we were discussing was the stream function and potential function. In particular, what I mentioned just very briefly at the end of last class is how to relate the difference in stream function values because along a streamline a stream function value is constant. Now, I have drawn streamlines which are having unique stream function values. Difference in that stream function value will give me the flow rate between the two lines.

So, in this case as far as this proof of this is concerned, I can see that here I have drawn two lines. One is here the streamline 1 and the streamline 2 let us say and they have their unique stream function value ψ and $\psi + d\psi$. What we and the flow that is taking place between these two streamlines because flow is taking place in this direction. So, the flow is taking place in this direction, and so, how much flow rate is? What would be the flow rate between these two streamlines? Similarly, there would be some other flow rate across there will be another third streamline there will be another streamline like this.

So, between these two vertical streamlines how much would be the dQ. Volume flow through an element ds of control surface of unit depth. So, I have picked up a ds which is an element length element ds of control surface and this has ds this has unit perpendicular to the screen. So, now $dQ = (\vec{V} \cdot \hat{n})\, dA$. So, this is the area element d this is the dA element ds multiplied by 1, 1 is that unit depth perpendicular to the screen that is the area and the area has a direction which is n hat that is the unit normal vector.



So, you are writing the flow rate as the velocity that you have and flow rate is velocity multiplied by the area over which the velocity is applicable. So, that means, v dot dA or v dot n hat dA n hat is the unit vector normal to this area. So, now, I write here v hat as u i hat + v j hat. So, this v the velocity vector v is u i hat + v j hat and u I am replacing by del ψ del y and v I am replacing by - del ψ del x this - goes there. So, this gives me the velocity v and then I have this n hat.

So, n hat is what would be the n hat? See n is this unit vector if we break it down into its respective components. So, n as a vector would be n sin θ i hat n as a vector if this angle is θ that means, you can show that this is this becomes n sin θ and this becomes n cos θ . How is it possible? This is the angle it makes as θ this is this angle is θ that means, this angle is θ, and this angle is θ means this angle would be 90 degrees - θ . Since, this is again 90 degree so, this angle would be θ . So, this dx would be then this dx would be ds cos θ and this dx would be ds cos θ and dy would be ds sin θ that is one thing that is there.

In fact, that is why we are writing you can see here sin θ becomes del y or it is del y del s and the cos θ becomes del x del s right because, this is the angle θ if this is a this angle is θ then this angle will also be θ . So, then this would be this ds so, sin θ would be dy by ds and cos θ would be dx by ds. So, that is what we have used. So, and if this is if this angle is θ that means, this angle is θ that means, this would be your n cos θ and this would be n sin θ . So, that is that is how it is I mean you take it further down.

So, this angle if this angle is θ then this angle is also θ and this angle will be 90 degree - θ . So, that is how $\vec{n} = |\vec{n}|\sin\theta\ \hat{\imath} - |\vec{n}|\cos\theta\ \hat{\jmath}$. So, you can see here that n can be written as n sin θ i hat and n cos θ j hat that means, and it is it is a - sign because, cos θ is y is taken as positive in this direction. So, y would be here negative y would be negative direction and then you have n as so, you have you have you so, you write that that is why you write as the magnitude of n sin θ i hat and magnitude n cos θ j hat and then then this sin θ and cos θ they are written as $\vec{n} = \frac{\partial y}{\partial s}\hat{\imath} - \frac{\partial x}{\partial s}\hat{\jmath}$ and since this is so, you want to write the write the unit vector n hat that would be the n vector divided by the magnitude of it. So, it would be this the same same expression goes there i del y del x j del x del s and the magnitude would be

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\left(\hat{\imath}\frac{\partial y}{\partial s} - \hat{\jmath}\frac{\partial x}{\partial s}\right)}{\sqrt{\left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial x}{\partial s}\right)^2}}$$

$$\hat{n} = \frac{\partial y}{\partial s}\hat{\imath} - \frac{\partial x}{\partial s}\hat{\jmath}$$

So, essentially the n hat that we are talking about here the n hat that we have here these n hat is same as so, so, these these goes there as n hat. So, you have these n hat as $\hat{\imath}\frac{\partial y}{\partial s} - \hat{\jmath}\frac{\partial x}{\partial s}$. So, now, you have this v dot n. So, we have taken care of v dot n and then multiplied by dA dA is essentially ds multiplied by 1 unit depth perpendicular to the screen.

So, you could have multiplied by 1 you write ds into 1 and so, now, you look at if you take the product i hat with i hat. So, you have i hat del ψ / del y if you if you have a dot product here if you have a dot product over. So, i hat dot i hat del ψ del y del y del s ds. So, what all are cancelling out del y is cancelling out with del sorry del y is cancelling out with del y del s is cancelling out with del del s. So, what you are left with is i hat del ψ del y dy because del s and this ds is cancelling out.

$$dQ = \left(\vec{V} \cdot \hat{n}\right) dA$$

$$dQ = \left[\left(\hat{\imath}\frac{\partial \psi}{\partial y} - \hat{\jmath}\frac{\partial \psi}{\partial x}\right)\left(\hat{\imath}\frac{\partial y}{\partial s} - \hat{\jmath}\frac{\partial x}{\partial s}\right)\right](ds)$$

$$\Longrightarrow dQ = \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx = d\psi$$

So, so, once again. So, it would be i hat del ψ del y and when you take the dot product. So, del ψ del y dy and this is giving you del ψ del x dx this ds will cancel out and so, you have dQ essentially is dQ essentially is linked to $\frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx$ and that is essentially this is equal to d ψ . So, you can see that the dQ is equal to d ψ the flow that takes place between 2 streamlines that is equal to that is equal to the difference between the 2 stream functions. So, this is the volume flow between any 2 streamlines in the flow field is equal to the change in stream function between those 2 streamlines.

So, that is the property stream function has. So, if you want to know q when here dQ is a differential flow we are talking about the 2 streamlines they are differing by stream function value which is a differential amount ψ and $\psi + d\psi$. If you do it over a finite difference if you have a if it is not a differential amount of difference. So, then it would be q between streamline 1 and 2 and then this whole thing v dot n dA would be then integral 1 to 2 then then dQ has to be there is an integration v dot dA there will be an integration and then that would be then by this exercise we note that v dot n dA is d ψ integration 1 to 2 which is equal to ψ 2 - ψ 1. So, for compressible so, this is a very unique property and this property can be utilized to draw the streamlines.

In fact, when we draw streamlines as I pointed out in the last lecture that you have a wooden log present and then you have the streamlines they are coming and taking a detour. And I said that the difference between the 2 stream function values if this is 1 meter cube per second per meter perpendicular to the screen and this is 3 meter cube per second per meter perpendicular to the screen. So, this difference will be the flow rate that is taking place that is the flow the difference between the 2 that gives you the flow rate that is taking place between the 2 streamlines. So, I said that if the 2 streamlines they are coming close to each other that means the same flow rate, but over a smaller cross sectional area. So, that means, the velocity increases if they are diverging that means, velocity decreases that is one property.

The other thing is it is expected that you will draw streamlines such that the stream function values that are equi-spaced. That means, if it is 1 it is 3 then the next one will not be 4, next one would be 5, next one would be 7, the other one would be 9 then that would be 11 all meter cube per second per meter perpendicular to the screen 7 meter cube per second per meter perpendicular to the screen. So, you make sure that this you intentionally you have this difference the value of stream function would be you will equi-space them as far as the stream function is concerned. So, if you do that then you will have a fair idea that the volumetric flow that is taking place between these 2 streamlines same volumetric flow is taking place between these 2 streamlines same here same here and same here. So, then you will have a fair idea in some places the stream functions they are converging, but you know that in the next series of streamlines near that place they are not converging that much, but you know for sure that the amount of flow that is taking place between each set of these adjacent streamlines they are same.

So, that visual will tell you how much what all I mean where the flow is increasing flow is decreasing and get a feel how the flow takes place. So, it is customary to draw streamlines by choosing the stream functions which are equi-spaced not arbitrary they are not chosen arbitrarily that they are put equi-spaced. So, with this background let us so, one thing is there that now if the flow is compressible if the flow is compressible then there is a way to work with these stream function and streamlines only thing is you have to bring the density inside these definition of stream function. So, for compressible flow at the steady state you can see that $\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial x}(\rho v)=0$, continuity equation was del u del x + del v del y equal to 0, but if you have a compressible flow you will have del del x of ρ u + I mean compressible, but steady state. So, so you do not have del ρ del t term.

So, $\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial x}(\rho v)=0$. So, in that case you have to define your stream function such that this $\rho u=\frac{\partial \psi}{\partial y};\ \rho v=-\frac{\partial \psi}{\partial x}$. So, so earlier it was u is equal to $\frac{\partial \psi}{\partial y}$ v is equal to - $\frac{\partial \psi}{\partial x}$ now it is ρ u and ρ v ρ you are putting in there in that scheme. So, now, when you instead of dQ now you will have $d\dot{m}$ instead of volumetric flow rate now you have to keep track of mass flow rate and the stream function the values that you have ψ and ψ + d ψ they are difference they would be in terms of mass flow rate. So, unit of ψ will not be meter cube per second per meter perpendicular to the screen kg per second per meter perpendicular to the screen.

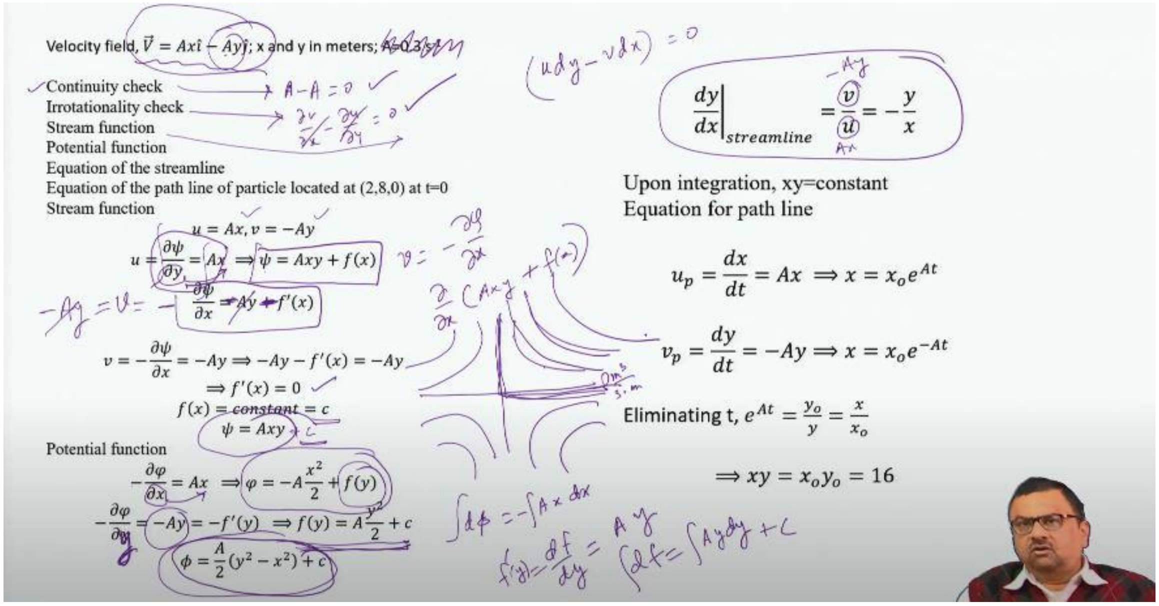
So, you have to make that arrangement because you are multiplying meter cube per second per meter perpendicular to the screen with a kg per meter cube. So, meter cube will cancel out and you have kg per second per meter perpendicular to the screen. So, once again this $d\dot{m}$ in just like you have dQ here

$$d\dot{m}=\rho(\vec{V}.\hat{n})\,dA=d\psi$$

$$\Rightarrow \dot{m}_{1\to 2}=\int_{1}^{2}\rho(\vec{V}.\hat{n})\,dA=\psi_2-\psi_1$$

So, this is a treatment one can have when the flow is compressible somehow density has to be brought in there you have a velocity field, this velocity field we had talked about earlier $\vec{V}=Ax\hat{\imath}-Ay\hat{\jmath}$ and let us say A is equal to 0.3 second inverse. So, as a matter of fact, it does not matter whether A is 0.3 second inverse or not. So, it does not matter. So, leave this out please. It does not matter x and y are in meters. Similarly, here it is. So, it is let us say if the velocity field is this way first you are asked to run continuity check.

Continuity check is del u del x + del v del y equal to 0 del u del x here u is A x. So, del u del x is del del x of A x that is A and - A y is v del v del y would be - A. So, A - A is equal to 0. So, continuity check is satisfied. Irrotationality check, irrotationality check is del v del x - del u del y equal to 0 del v del x here is v does not have any x variable u does not have any y variable.



So, naturally these two do not exist. So, it is irrotational also. So, continuity and irrotational valid means now you can have both stream function and potential function. Stream function how do you find stream function? Stream function here you can see u is equal to A x and v is equal to - A y that is what the velocity field is all about. So, u is equal to del ψ del y and that we are writing as A x.

So, if you if you take this part and do the integration. So, del y goes to that side. So, it would be this del y will go to the right hand side and then when you do this integration. So, this side it would be ψ left hand side would be ψ and right hand side would be A x d y integration. So, that would be equal to A x would be outside the integral.

So, it would be A x y + constant of integration, but here constant of integration would be a function of x because it is a partial derivative. So, constant of integration would be a function of x. So, you can write $\psi = Axy + f(x)$, but you need to understand you need to you need to find out this functional form F x. So, I have another variable there v is equal to - A y and what is v? v is v would be $v = -\frac{\partial\psi}{\partial x}$ and here del ψ del x is what? $\frac{\partial\psi}{\partial x}$ you are taking derivative of ψ with respect to x. So, $\frac{\partial\psi}{\partial x}$ is A into y that we can see here del del x of A x y del del x of A x y would be simply A y right del del x of A x y + F x that is what we are looking at.

So, del del x of A x y would be A y and del del x of F x would be $f'(x)$. So, that is del ψ del x and then the v is equal to - del ψ del x. So, v is equal to - del ψ del x means v is equal to - del ψ del x. So, this - if I put a - sign then it will be - A y and - $f'(x)$ and then that would be equal to the actual v that we have. So, that is equal to v and v in this case is - A y.

So, I can see here - A y is equal to on this side I have - A y - $f'(x)$. So, what that means is - A y and - A y will go out and $f'(x)$ becomes equal to 0 that is exactly what we see here $f'(x)$ is equal to 0 and $f'(x)$ equal to 0 means $f(x)$ is equal to constant. So, now, I can treat this as a truly a constant because of this exercise $f(x)$ is a constant. So, $f(x)$ is a constant means now we can write this ψ as A x y + some constant. So, ψ it be A x y + some constant.

Now, we can it would be my choice what constant I assume because I remember that when we when we have drawn this we said that if you try to plot these lines these would be the these would be in the 4 quadrants this would be the streamlines if you plot this A x i hat - A y j hat. So, then what would be that constant? Now, who defines this constant? I mean I this I said that I say that this ψ is having some value A x y + constant. Now, I have one streamline that passes through 0 0 this is that streamline this is the streamline that passes through 0 0. So, the streamline that passes through 0 0 if I assume that that streamline that that stream function corresponding to a streamline that passes through 0 0 that value of stream function is 0. That means, when x equal to 0 we are supposed to have a c, the constant of integration, but if I assume that the ψ is 0 I must have a reference ψ because here if you if you look at it what we are interested in is how ψ is changing the delta ψ matters to me.

So, it is the ψ there has to be a reference ψ . So, if I assume that the ψ value would be 0 for the streamline that passes through origin that is x equal to 0 y equal to 0. So, that means, I am I can I can assume that c to be 0 in that case that that is my that is how I set my reference. So, that means, this streamline will have a stream function value of 0 meter cube per second per meter perpendicular to the screen. Then this would be then 3, 5 something then this would be, but these have to be equi spaced the ψ value they have to be equi spaced ψ values that are that that has to be there.

And similarly, if we try to find out what is the potential function that same exercise similar exercise here the potential function I will write here

$$-\frac{\partial \varphi}{\partial x} = Ax$$

$$-\frac{\partial \varphi}{\partial x} = -Ay$$

So, this again you do the integration in a very similar way. So, you get φ is equal to - a x square by 2 + f y mind it is it is the φ this is del φ del x a x. So, that is that is integration of d φ that is equal to the - sign goes there goes there and this d x goes to the right hand side.

So, you have integration integration d φ that would be integration of a x d x. So, that is why we have a x square + 2 there is a - sign outside and then there would be a constant of integration, but since it is a partial derivative that constant of integration would be a function of y. So, now, you then take a so, this is your φ. So, this φ, you take the derivative with respect to y this should be y. So, this should be y take a take the derivative with respect to y and then equate that with - A y that that is what the v is.

$$\frac{\partial \varphi}{\partial x} = Ax \implies = -A\frac{x^2}{2} + f(y)$$

$$-\frac{\partial \varphi}{\partial x} = -Ay = -f'(y) \implies f(y) = A\frac{y^2}{2} + c$$

$$\phi = \frac{A}{2}(y^2 - x^2) + c$$

So, - del φ del y if you want to do if you take a derivative of this with respect to y. So, then this derivative of this with respect to y would be 0 because there is no y parameter and this would become f prime y. So, $-f'(y)$ has to be equal to - of a y that is that is what the velocity is all about in the top. So, so, it so that means, if $f'(y)$ means dF dy if dF dy is equal to a y that means, dF $f'(y)$ is equal to dF dy and that is equal to a y. So, what that means is then dF integration that would be equal to integration A y dy and + some constant of integration.

So, now, you have this would be $A\frac{y^2}{2}$ when you do the integration. So, that is why f y is equal to $A\frac{y^2}{2}$ + constant of integration. So, so, φ can be written as this and then you have your choice of what constant you choose what is your reference and it would be prudent to use it in a more ingenuous way so that the c can be put to 0. So, so that these are these are some of the things which you so, stream function and potential function if you have a velocity field you can immediately find out stream function and potential function. And any of these whether it is the stream function whether it is the stream function that you have ψ is equal to a x y or potential function φ is equal to this.

So, these are unique parameter stream function moment it is defined that gives you both u and v you can derive from there. So, these are some of the important aspects of streamlines streamlines and stream functions, but at the same time you can you can see that. So, what would be the equation of a streamline equation of streamline would be $\left.\frac{dy}{dx}\right|_{streamline} = \frac{v}{u} = -\frac{y}{x}$. What we what we had written what was our definition of streamline u dy - v dx that we had a k hat outside that is equal to 0.

So, dy/dx along streamline is $\frac{v}{u}$. So, v by u in this case is simply v by u in this case is simply - y by x v is - a y and u is a x. So, v is - a y and this is a x. So, that is why we have this dy/dx for the streamline the slope of the streamline is - y by x. So, you can see whether the slope of the streamlines are truly - y by x or not that that you can check here. That is one thing and if you are trying to find out what is the equation for a streamline equation for the streamline would be here you can see that the equation of the streamline is x y is equal to constant because if φ is if ψ is equal to a x y that is the equation of stream function then you can see that the stream function is a constant value along a streamline.

So, along a streamline a x y is equal to constant or in other words x y is equal to constant x y is equal to constant if you go to coordinate geometry and try to find out what form it gives it gives me the rectangular hyperbola. So, that is so these are classical x y is equal to c lines. So, this is one thing which we can see immediately from here. And here that is one thing and the other point here is that we may like to see in this case what would be the equation for path line, but that is also another area which we need to look into. At this point I think we should we need to we need to next we have to focus more on the complex potential, but more or less we got a feel for what stream function is and what potential function is though even I think I have not made it I have not explicitly mentioned what would be the role of stream function and potential function together, but they are at least as far as the stream function is concerned and how the how it helps understanding the streamlines and how it understand how it helps in getting visual of the entire flow process that itself is clear and if you at this point if I give the stream function you should be able to find out the velocity field or if I give you the velocity field you can find out the stream function same applies for potential function and we have more or less got a feel how streamlines and potential lines they are oriented.

I think I must stop at this point what I will do next in the next class is I am next class I will be focusing more on complex potential, but I would be a little bit of this potential function I need to discuss before probably spend couple of minutes further on these potential functions and then I will move to complex potential and I will see that how the complex potential helps us in solving cases where we have very unique type of unique situations evolving for fluid flow. For example, you may have a vortex running and then on top of that you have another uniform flow taking place. So, if you have a combination of these two how will we find out the velocity how will I find out the stream how streamlines will be located in vortex we know that it is circling and the flow which is uniform. So, if you have a combination of these how the streamlines will orient themselves and then how you find velocity at different points. So, these are the unique things because we are not doing stream just having a velocity vector and finding out stream function that is not objective.

Objective is to have a complex flow problem and there we try to come up with the streamlines at least from the theory the stream function and streamlines and from there we want to extract the information what would be the velocity where streamlines are converging where they are diverging when you have complex flow like this. So, that is something which we are heading to in the next lecture the title would be complex flow. So, there we will be utilizing this stream function and potential function these concepts that we have talked so far. That is all as far as this lecture module is concerned. Thank you for your attention.