Momentum Transfer in Fluids

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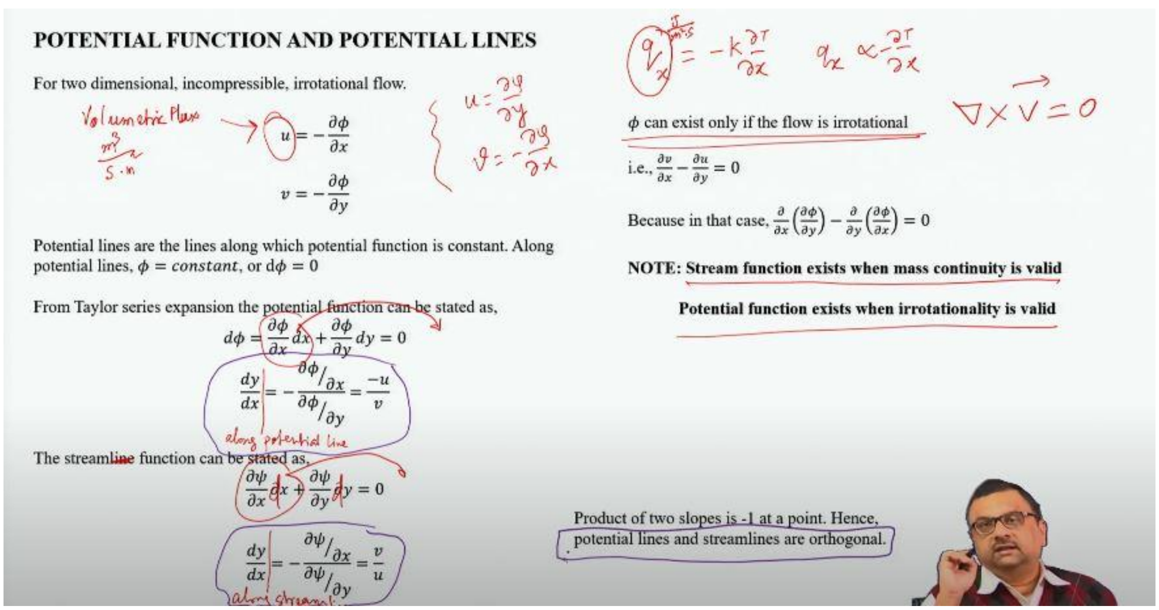
Week-05

Lecture-24

I welcome you to this lecture on momentum transfer in fluids. In particular, we are going to discuss today about stream function and potential function. At the end of last class, we had a very brief introduction of what stream function can or rather what stream lines can do, what advantage stream lines can bring in bring to the table. So, when the stream lines will be connected to stream function and we will discuss these theories in today's lecture. So, what we have here is we discussed briefly we introduced this in the last class you we talked about something called stream line and path line. We discussed stream line and path line, and we mentioned that if we try to track the fluid particles as they travel, if we put some tracer, we put some paper boat on a flowing stream, and we see where the paper boat is traveling.



So, essentially, we have drawn a picture. You may recall that there is a wooden log present, and then fluid that is going, it has to bypass. So, because of that you will find that if you put a put some paper boats they will they will be taking some detour. So, if we try to track these lines that lines that the paper boat will follow. So, then that gives us some information and in fact, how that gives us really useful information that we are going to discuss down the line.

And we have made a we have differentiated what is path line and what is stream line. Essentially, if you trace the line that the paper boat will follow in a flowing stream that line is path line whereas, a stream line has a more rigorous definition stream line says that this is the line on which if you draw a tangent at any point that tangent will have a tangent will merge with the velocity vector at that at that particular point. So, that means, if this is this is the streamline as you can see if this is the streamline, then if you draw a tangent at this point and the velocity direction of the velocity vector velocity field at that particular point would be the same, these two directions are the same. So, based on these criteria we said that if we pick up a dr as an element of length along a stream line. So, if dr is an element of length along a stream line and if we expect that the velocity field and a dr will have the same direction then $\vec{V} \times d\vec{r} = 0$.

And we have continued with this exercise and we end up with this expression which we said that this is the expression for stream line and we mentioned that these streamlines and path lines they will converge they will they will fall on each other when the flow is steady, but if the flow is unsteady that is not the case. So, then what we said is that since these, since the velocity components u and v, they have to follow the continuity equation that is mass conservation that it has to obey. So, a new function was defined which is referred as stream function and the idea of putting a stream function is that the only stream function can handle both the velocity fields you do not have to specify u and v instead only ψ can define both the velocity components in the x and y directions. So, what is that? That is the way it is defined is

$$u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x}$$

So, here we have since the continuity has to be satisfied we note that if we make a definition like this then the continuity is satisfied.

So, that continuity your mass conservation permits the use of this definition of stream function and on top of that we have seen that since v dx is equal to u dy that is that is what that is what the observation here. So, if v dx and u dy so, $u\,dy - v\,dx = 0$. So, in that case if we put instead of u if we put del ψ del y and instead of v if we put - of $\frac{\partial \psi}{\partial x}$. So, that - and - that will form +. So, if instead of v we put $\frac{\partial \psi}{\partial x}$

So, then it becomes $\frac{\partial \psi}{\partial x}$ dx and $\frac{\partial \psi}{\partial y}$ dy that becomes equal to 0 that is that is what it boils down to. So, in that case it is we can write this as

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$d\psi = 0$$

So, this is the condition which is valid along a streamline because a stream definition of streamline is u dy - v dx equal to 0. So, what this means is along a streamline d ψ is equal to 0.

So, what that means is along a streamline ψ is constant. So, that is a very unique feature that stream function brings to the table that is you have a very you have a value of ψ is equal to ψ 1, ψ is equal to ψ 2 these are all constant some numbers. Let me tell you if you want to dig into what is the unit of ψ you will see that ψ would be what is ψ by the way ψ is del ψ del y is equal to u and u has a unit of meter per second. So, it is ψ typically the ψ will have unit of you will see meter cube per second per meter perpendicular to the screen. That means, when I draw a line I am assuming that unit depth perpendicular to the screen that this line is the line represents all such flows that is there unit depth perpendicular to the screen.

That means, this line is valid for you just like you have when you write a velocity at r or velocity at x you that is that velocity is valid for x and x + dx within that the velocity is valid. Similarly, here the streamline you are assuming that streamline is represents the flow for unit as you see on the screen, but that is valid for unit depth perpendicular to the screen. So, the unit of stream function is meter cube per second per meter perpendicular to the board perpendicular to the screen. So, this is what it is and so, this is consistent with the if you put u as meter per second and all these and y as meter you will get there. So, because it is finally, it is meter square per second if meter cube per meter if you write.

So, these are some numbers. So, ψ would be maybe 1 meter cube per second per meter perpendicular to the screen this ψ is 2 meter cube per second per meter perpendicular to the screen like this. So, these ψ 's are numbers. So, these streamline represents a unique value of stream function single value of stream function. So, now that is very important.

So, not only the ψ you have defined the ψ is consistent in terms of mass conservation that is continuity equation ψ gives me a single variable using that I can find out what is u and what is b I do not need to have two different expressions for u and v that is the other thing, but most important thing is if you want to draw streamlines these were the value of stream function along a streamline will remain constant. So, how we can extract this information. In this context I must tell you something called a potential function also just like we mentioned stream function we there is a very similar function which is known as the potential function and the potential function is defined here as the ϕ is the potential function. So,

$$u = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y}$$

So, what was the stream function? In stream function we said

$$u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x}$$

So, this is u and v. So, here in this case we are writing u and v in terms of potential function you can see the change here it is u is $\frac{\partial \psi}{\partial y}$ here u is - del ϕ del x v is - del ψ del x, but here v is - del ϕ del y. Fact is that this potential function you may see first of all why do I need suddenly another function because stream function was good enough. The fact is that potential whenever there is a some cause and effect relationship you always have a driving force and because of the driving force there is certain changes happening.

For example, when you see a current flowing through a conductor you have a voltage gradient $\frac{\partial v}{\partial x}$ and because that is the cause and what is the effect you see current. When you see a heat flowing there is a temperature gradient $\frac{\partial T}{\partial x}$ and you what you will see is there is a heat flux arising from this. So, if you look at let us say the heat flux if you look at q x is equal to - k dT dx you might have what if you write it in terms of derivative it is - k del T del x. So, you can see here that the q x you can write so q x is a heat flux. Heat flux means joule per meter square second.

So, you can write this u also has the volumetric flux you can you call u the velocity you can write this as volumetric flux. Why volumetric flux? That means meter cube per second volumetric flow rate meter cube per second per unit area that is the definition of flux meter square. So, that gives you meter per second. So, call u and v as a volumetric flux in x and y direction. So, similarly here heat flux.

So, heat flux is proportional to - del T del x. Similarly, you will have mass when you have a diffusion happening you will find that the mass flux would be proportional to concentration gradient just like you have current flowing there would be proportional to voltage gradient. By that token we were looking for some parameter which we can write we can relate in a very similar manner we should be having another parameter and that is what is referred here as potential function. So, velocity is the effect and the cause is the potential gradient. Now, the fact is this potential function and pressure they are very similar you can see you might have said that ok I can I not write that there is a velocity because there was a pressure gradient that is as simple as this.

So, that is also cause and effect relationship and we could have been happy with that, but the fact is that when it comes to these temperature etcetera. So, they were defined that way that if when you have a higher temperature and a lower temperature and there is a temperature gradient existing then you have a heat flux. So, temperature is defined that way your voltage and current they are linked that way, but the pressure mind it pressure is not defined that way pressure is defined as - one third sigma xx + sigma yy + sigma zz the three normal stresses average of that and put a negative sign. So, that is what is defined how that is how pressure is defined. Pressure is not defined as something by because of which you have flow will have higher pressure to lower pressure.

So, there is a there is a certain difference. So, that is that pressure has already been defined in a different way. So, one must say that I need another parameter which is defined truly I mean whose driving force will cause the velocity to be the velocity to be registered. So, in that in so, that way potential function becomes important. In fact, this potential function will lead to again a set of so, called potential lines just like we have seen stream lines we will also see something called a potential lines.

So, potential lines once again are the lines along which the potential function is constant. So, along potential lines,

$\phi = \text{constant}$, or $d\phi = 0$

So, once again you have this

$$d\phi|_{\substack{\text{along potential} \\ \text{line}}} = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

If you have if you have d ϕ to be 0 then the potential function what would be the slope of the potential function. Before we get to that I must note here that the ϕ can exist only if the flow is irrotational that is one major condition irrotational means curl of the velocity field that is equal to 0. So, that is what is the condition for irrotationality.

So, that is you have that means $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0$$

So, this condition is satisfied. So, so what important thing is stream function exists stream function make sure that a continuity satisfied that is that is what we mentioned here. We noted that stream function satisfies the mass conservation stream function satisfies the continuity equation, but potential function on the other hand we have to make sure the potential function will only exist. So, stream function does not require this irrotationality condition to be in place stream function only requires that the continuity is satisfied that means flow is feasible. Whereas in case of potential function not only the flow has to be feasible on top of that flow has to be irrotational then only a potential function exists otherwise potential function does not exist that is one condition. So, stream function exists when mass continuity is valid potential function exists when irrotationality is valid and on top of that mass continuity of course, has to be otherwise flow is not feasible mass continuity has to be valid everywhere.

Now, if we try to see what how the potential lines will look like I mean if we come up with this type of potential function and if we try to find out how this potential lines will look like. So, let us see what would be the slope of the potential lines we see from Taylor series we see that this d ϕ can be if the definition of potential lines is lines along which the potential function is constant stream lines was defined the lines on which we draw tangent and we see that the tangent follows the direction of the velocity at that particular point that is that is the definition of stream function and it just so happened that stream lines are the lines along which the stream function was constant. Here potential lines we say that potential lines are the lines along which the potential function is constant. So, we have $\partial\phi = 0$ because ϕ is constant and then we write this and then we take one to the if we take this one to the to this side as $-\frac{\partial\phi}{\partial x}$. So, then dy/dx from here from this equation we will see that the

$$\frac{dy}{dx} = -\frac{\partial\phi/_{\partial x}}{\partial\phi/_{\partial y}} = \frac{-u}{v}$$

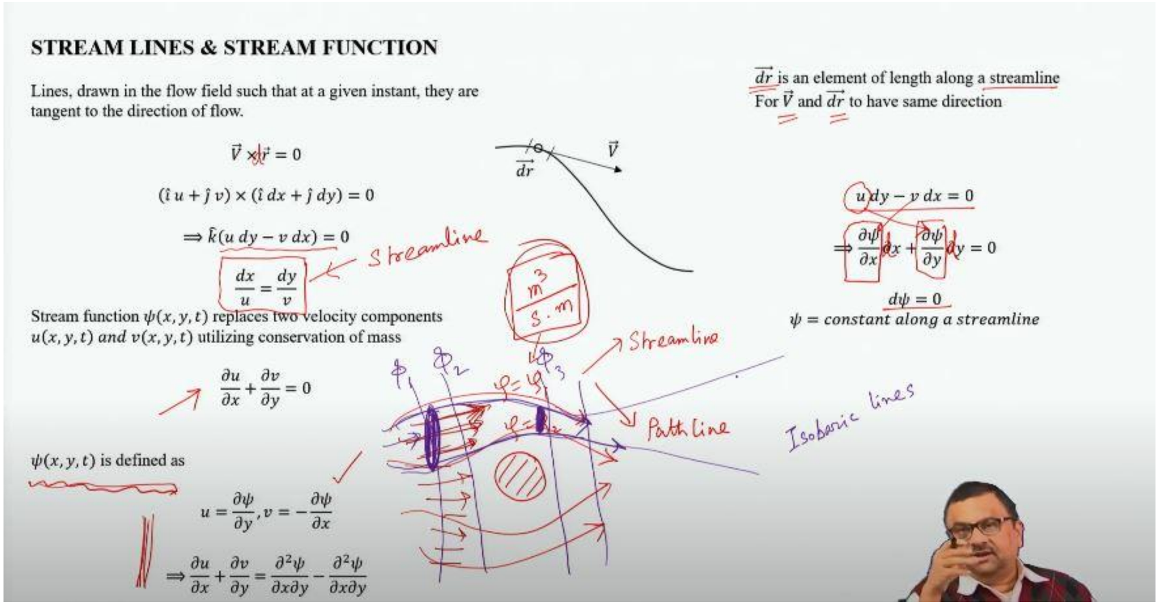
If we look at the same thing stream streamline for that is that is basically the streamline the stream function the you call it stream function can be stated as $\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$. So, here dy/dx along streamlines dy/dx along streamline that would be equal to - of I mean if we if we take once again these term to the right hand side with the - sign and then take dy/dx that would be $-\frac{\partial\phi/_{\partial x}}{\partial\phi/_{\partial y}}$ and then del ψ del x is - v and del ψ del y is u. So, this - and - we cancel that it becomes +.

So, dy/dx along streamline is v by u. So, you see here that dy we see here that dy/dx along potential line is - v by u and dy/dx along streamline is - u by v and this is v by u. If I take the product of these two that means, dy/dx along potential line multiplied by dy/dx along streamline we see that - u by v into v by u. So, the product would be u and v they will cancel out and product would be - 1. So, if the two slopes the product of the two slopes is - 1. So, from coordinate geometry we can say that the two lines are mutually perpendicular they are orthogonal to each other.

So, the important observation here is potential lines and streamlines they are orthogonal. So, what that means is the potential lines. If these were the streamlines in the last slide, then the potential line would be orthogonal to these lines. So, that means these are the potential lines. So, these are the potential lines, and you have, in fact, there is another line also one refers which is called the isobaric line. Isobaric lines isobaric is where the pressure remains constant.

So, you will find that the isobaric lines though they will also follow the same line because not only so, this corresponds to let us say value of ϕ 1 this corresponds to ϕ 2 this potential function is constant. Isobaric lines will also be a very similar line and they will also have isobaric means the pressure is constant along that line. So, it would be p 1 p 2 p 3 and that makes perfect sense because flow is taking place from left to right. So, flow is taking place from left to right means pressure is higher on the left side pressure is lower on the right hand side and there is a flow taking place. So, you can expect that the pressure will vary in this direction, but along the cross section pressure is remaining same, but pressure is changing as you move from left to right.

So, this is so, potential lines and isobaric lines they are synonymous. However, we have to for the sake of correctness we must have another function. It is not just that for the I mean we want to be very rigorous and so, that is why we have come up with this potential function. Fact is you will see that down the line we will come up with something called a complex potential which is defined as F (z) where F (z) where z is we write this as x + i y since x and y in this system x and y they are mutually perpendicular so, we write a complex number z as x + i y and a complex potential we will define since ϕ and ψ they are the streamlines and the potential lines they are orthogonal to each other we want to extract that trait that character to write a complex potential as $\phi + i \psi$ a complex number and this is referred as complex potential and this complex potential helps us in doing superpositions and all. So, this complex potential also and a potential function also has a role there.



So, it is not just that we when pressure was defined in a different way. So, that is why we are getting into this it is not correct. We have a specific purpose to this potential function. So, we will get to that down the line. Before I move from this slide another point I must emphasize is that some I mean I have found one or two books which put this u as - del ϕ del x v as - del ϕ del y and the reason is that since it is cause and effect relationship and generally the flux is in the direction of negative gradient.

So, similarly velocity should be in the direction of negative potential gradient. I found at least in one book I saw that there is a - sign, but there are several books I have seen where it is not - rather it is put as +, u is simply del ϕ del x v is simply del ϕ del y. It is just the question of choice and the final results does not get affected if you take both because both of them you are not doing preferentially one - another +. If both are considered + then the potential function value would be the sign would be reversed, but that does not affect our calculations down the line. So, later on when you go to complex potential and all so at that time we may simply follow the + sign because most of the books they follow the + sign.

It does not matter when it comes to the final calculation that we are trying to get out of this. Now, there is I mean we are particularly fascinated with these concept of streamlines because streamlines gives me the real visual of where the flow takes place. Potential lines are fine I mean they are sort of isobaric lines and tell me where all the pressure is constant. They have their merits, but streamlines are something which we are particularly we want to we have to get to. Now, here you can see streamlines and stream function.

The stream function has another unique property what is that? Let us say this is a control surface. You need depth into the screen this is known as a control surface. You need depth into the screen and this is one streamline this is one streamline and this is another streamline. So, this is let us say one streamline has value of stream function as ψ and another streamline has a value of stream function as ψ + d ψ .

So, these are the two different streamlines they have their unique stream function value. If we try to find out how much flow takes place between the two streamlines? We want to know how much flow takes place between these two streamlines. So, how much flow takes place between these two streamlines because streamline unit is meter cube per second per meter perpendicular to the screen. So, what meter cube per second? What flow? Flow is taking place between these two streamlines. How much is taking place between these two two streamlines, how much flow is taking place between these two streamlines, we want to know that.

So, let us say I am writing it as dQ between these two streamlines the flow is dq. So, what we will do is we mean these in this calculation in these expressions where we what we are going to prove that the if this is ψ and if this is having ψ and this is having ψ + d ψ then one can write this dQ is that flow that is taking place the flow rate dQ and this d ψ they are equal that what that means is the amount of flow that takes place that means the flow that is taking place between the two streamlines that is equal to ψ 1 - ψ 2 this difference in the stream function. So, what is the implication do you do you see the immediate implication here, I see at some point the two streamlines they are coming close to each other. I see that this is the streamline and this is the streamline, this streamline is having this much of difference here this much of gap was there here the gap is reduced. If I try to tell you that the volumetric flow rate between these two streamlines that remains same.

So, in that case you will see that the volumetric flow rate that is taking place over this area has to be same as the volumetric flow rate occurring in over this area. So, what that means is these two streamlines as if this forms a conduit. So, here I see that the cross sectional area of the conduit is decreasing. So, what will happen to the velocity same q divided by area, same Q divided by the area. So, here the area is more velocity is less here the area is less velocity would be more.

So, in that case I will see that a flow takes place here the velocity of the fluid is increasing and at some point maybe it is diverging. So, we know that in those areas the velocity is decreasing. So, by looking at the streamlines and when we see the streamlines, they are coming close to each other, I know that there the velocity is increasing. So, streamlines are not only giving me a visual of where the paper boat will travel also the streamline will give me the give me an idea where the velocity is increasing and when where the velocity is remaining constant, remain constant means they will remain parallel to each other the two streamlines or they are diverging means the velocity is decreasing.

So, streamlines they have these unique properties we will go through this derivation in our next lecture.

Till that time let us take I mean till that time you go through this go through this definition of streamlines and stream functions and try to appreciate what all these streamlines what all what all visuals what all various features streamlines can illuminate when it comes to the when it comes to this flow process. That is all as far as this lecture module is concerned. Thank you for your attention.