Momentum Transfer in Fluids
Prof. Somenath Ganguly
Department of Chemical Engineering
IIT Kharagpur
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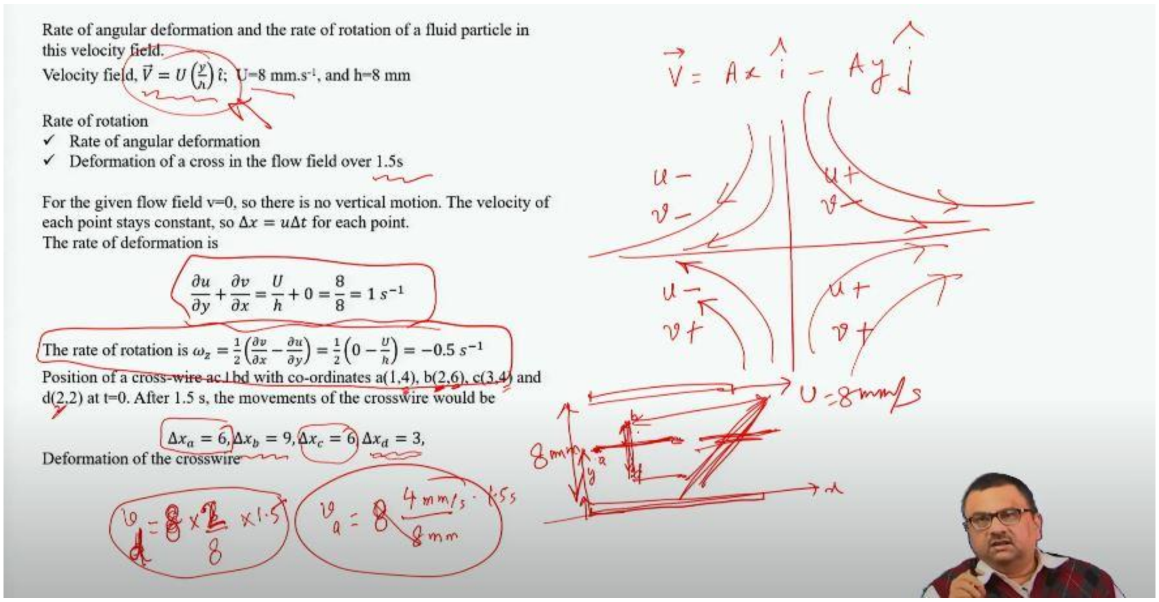
Lecture-23

I welcome you all to the lecture of this Momentum Transfer in Fluid. We were discussing about motion of fluid particles. In particular, we were discussing about some conditions of rotationality and angular deformation. So, that we know which one is inviscid flow and which one is a viscous flow. So, there we had at the end of last class, I gave you two examples.



One was that the velocity is Ax, we have this velocity as we gave an example of Ax i hat minus Ay j hat and I mentioned in the class that you may like to think about this we found was the flow is irrotational there is no rotation involved. So, what exactly what type of flow we are talking about. So, you can see if we go by this if we create a quadrant here. So, here we know that both x and y both are positive here.

Here we have x as negative and y as positive, here both x and y both are negative and here we have x positive, but y negative. So, in that case, we have u is equal to Ax. So, x positive y positive that means, in this quadrant u is plus v is minus because there is a minus sign. In this quadrant x is negative. So, u becomes minus and v becomes also negative because y is positive and so, y positive means the velocity becomes negative.

Here, in this case, both are negative, so that means you have u as x is negative, which means u is negative, and y is negative, which means the v becomes positive in this case. So, here v is plus and here in this in this quadrant I see that x is positive. So, u is positive and v is negative. So, negative and minus the product would be positive. So, we are talking about a flow which is in which is positive here, and u is positive here, and v is negative here.

We will see down the line that is this I mean if you can even plot this and see what kind of velocities at various positions of x and y what kind of velocities this can give, but if you allow me, I can tell you u is positive means the x component is positive, but y component is negative. So, flow will be in this direction x is increasing in a positive way, but y is decreasing. So, we will see that the velocities would be some flow would be something like this here the flow would be something like this. Here in this quadrant you will see the flow would be u negative u negative. So, moving in the negative direction, but v positive.

So, u will go something like this and here it is both u positive and v positive. So, both are moving in the positive direction. So, you will have this. So, you we were we are looking at I mean if you if you care to take some values of x and y find out for a as I said one second inverse and try to plot velocity at various locations and see what we end up with, you will end up with this type of a flow. So, this is this flow we are saying that this flow they does not have any rotational component.

Anyway, so what we will do is I said that when there is one upper plate moving at a velocity capital U and the lower plate is fixed and we try to find out whether there is rotation or angular deformation involved. So, I in fact, we already we have seen that the velocity profile can be written as this. So, if we if we work with this same system that is lower plate is fixed and upper plate is moving at a velocity u. u is here 8 millimeter per second and this distance is 8 millimeter and this is 8 millimeter per second and if I take if I am if I get a cross here a, b, c and d, a is a location of a is 1, that means, x component is 1 and y component is 4. If this y starts from here, so naturally you can see if this is your y and this is your x.

So, x 1 and y 4, so it is halfway because total distance is 8 millimeter out of that the y coordinate is 4. So, my point a is here. So, this is point a, point b is 2, 6, 2 is here, 2, 6, 2, 2 is here and here it is. So, this is this a is 1, 4, b is I have gone 1 and then 1, 2 and here I have gone all the way to 4, 5, 6. So, that means, this distance is this distance is 2 millimeter.

So, this is the b, c is 3, 4. So, y remains same as 4 here. So, this is c, the y value remains same as 4, x value becomes 3. So, this was 1 here and this is 3 here. So, c becomes 3, 4.

So, this distance is 2 millimeter. So, this distance is 2 millimeter, this distance is let us say this is c and d is 2, 2. So, that means you have 2 here, b has the same 2, and the x component is 2 as this. So, it would be on the same line as b and y value is 2, c this is 4, this point a is located at y value of 4. So, this is located at y value of 2.

So, that means, this distance is 4, 2 millimeter, 2 millimeter and this distance is 2 millimeter, these are not to scale actually, this is not exactly to scale. So, let us say I am taking this cross wire and I am trying to track this what happens after let us say 1.5 second. So, we note here that the velocity here would be $\vec{V} = U\left(\frac{y}{h}\right)\hat{\imath}$. So, you have the y is here in this case what would be the velocity v a? v a would be equal to velocity u into y is 4 millimeter and h is 8 millimeter and u itself is given as 8 millimeter per second.

So, you put these as 8. So, v a would be equal to 8 by 8. So, this is 4 millimeter per second. So, that that would be your v a 4 millimeter per second.

So, you in 1.5 seconds you will travel the point a will travel in x direction because it is only the x velocity we are considering here only the x component of the velocity is existing. So, we have v a the total Δx_a that is the point a would travel by a distance which is equal to 4 millimeter per second into 1.5 seconds. So, over 1.5 seconds 4 into 1.5 is 6. So, we note that delta x is 6 millimeter. So, we note that x a will move by 6 millimeter. So, whatever is its position is it was x component was 1.

So, that after 1.5 second this x component would be 1 plus 6 that is 7. And similarly the c will also move because this is only a function of y. So, c will also move at the same velocity. So, c will move to a distance by a distance 6 millimeter again. So, c initially c was 3 comma 4.

So, now, it would be 3 plus 6 which is 9 comma 4 that is a new position. And what would be the position of b and d? You can see that the position of b would be first of all what is the point b would be moving in x direction at a velocity 8 millimeter per second y is what y is 6 millimeter right b had a y value of 6 millimeter. So, 8 into 6 is the y divided by 8. So, that is that is so, it is moving at 6 millimeter per second into 1.5 seconds. So, that is 9 6 into 1.5 that gives me 9. So, Δx_b is equal to 9 and Δx_d in this case the y value is simply y value is simply 2. So, 2 if you put instead of 6 you put 2 if you are looking at v d if you are looking at v d. So, v d would be simply instead of 6 I will put 2.

So, it would be 2 into 1.5. So, it would be Δx_d would be 3. So, now, you look at now you find the cross wire positions you can see here that the point a and point c they have moved in x direction by 6 and 6 respectively. So, you expect I mean it will remain perpendicular to each other if point b also moves by 6 millimeter and point d also moves by 6 millimeter, but how much are they moving? Point b is moving by 9 millimeters all the way. So, you can see now the revised position is probably here like this because this has moved by 9 millimeter whereas, this point has moved only by 3 millimeter.

So, 3 times this. So, you can see the cross wire that you started which was mutually perpendicular to the each other I mean that is what that is how the coordinates were chosen after 1.5 second under this flow field we end up with the cross wire having a shape like this. So, obviously, this cross wire has undergone rotation and or angular deformation. So, how to find that? Quickly what we will do is if we try to find out the rate of rotation we will write this omega z because the ω_z is the only thing existing we are looking at xy plane deformation in xy plane. So, ω_z is existing other omegas are not existing is

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

And $\frac{\partial v}{\partial x}$ does not exist.

So, naturally $\frac{\partial v}{\partial x}$ is 0 and del u del y is simply u by h. So, we have u is 8 millimeter per second h is 8 millimeter. So, this becomes minus 0.5 second inverse that is the rate of rotation and if someone wants to find out what is the rate of angular deformation then it would be instead of minus this would be plus. So, that is why we have plus here and the half will not be there.

So, that is why we have instead of 0.5 we have 1 and instead of minus we have plus. So, this becomes the angular deformation this becomes the rotation. So, by looking at the velocity field itself this is the velocity field this is that from Eulerian approach we have come up with that velocity field. So, this velocity field by delving into it a little bit by poking it a little bit we will immediately find out whether there is any rotation involved whether that is del cross v is 0 or non-zero and then we can immediately find out what is the rate of angular deformation and what is the rate of rotation.

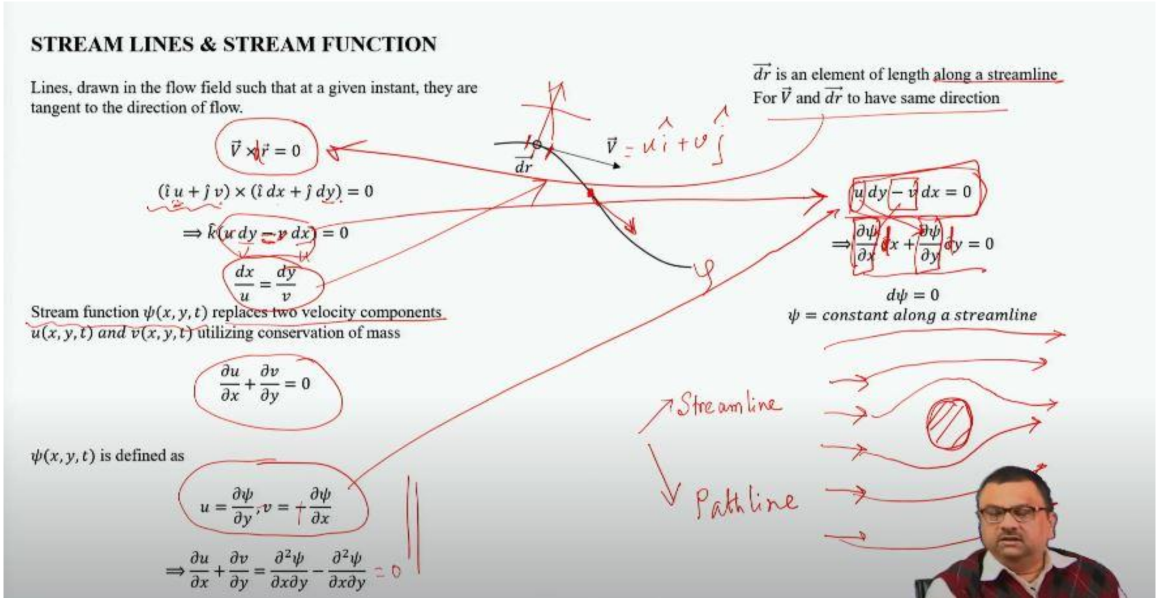
So, these are the few things which one wants to know suppose I am suspending in this in this fluid in this motion and then I want to know how much twist this that small element will undergo. So, there are, of course, that will distort the velocity field, but the fact is that these are some of the things some of inherent some of essential characteristics of fluid flow that we must be aware of when we try to work with the inviscid and viscous flow. So, this is one thing which we need to know. Another point is you may recall we talked about translation, we talked about rotation, we talked about angular deformation, and there was a fourth category which was linear deformation, and linear deformation was where I started with a square and ended up with a rectangle. So, how do we characterize linear deformation? The fact is there is something called a volume dilation ratio one may like to check and if this is exactly same as the continuity equation and if you see that the volume dilation ratio is 0 then that means, that the area of that square that you started with and the area of the rectangle that you ended up with those two areas will be same.

Fluid Deformation (Linear deformation)

Volume deformation rate,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot\vec{V}$$

So, that means, the volume deformation rate is 0. So, volume deformation rate is given by this quantity. So, these are the four types of flows four types of motion of fluid particles that are possible and most likely you will have all or maybe just the translation may not be the rotation and all those. So, this would be a mixture of all these elements of flow. What I will do is I in fact, that is where I am heading to in my next series of lectures in this course is we will define something called a stream function because you may see that I mentioned about that movement in a circular path and there I mentioned about a streamline.



So, this may be at this point this is we need to clarify this point what exactly the movement in a circular line means. In the next lecture we will have a detailed treatment of streamlines and stream functions and how it relates to the overall velocity field and everything, but here I try to give you some glimpses of it. One is that there are various lines possible when it comes to defining a flow field. There are two lines that are that are quite common in this regard. One is referred to as streamline, and the other is pathline. There are a few other lines also, but I think streamline and pathline this needs to be understood in greater detail.

Pathline is a line followed by a fluid particle during its course. I mean, you can have a tracer introduced in the flow path, and you will see where that flow is going. For example, you may have some small color added and you see where the color is traveling or you may have a paper boat and see where it is traveling and if you connect those points and that gives you a path. So, you will find that suppose there is an obstruction and the fluid is traveling like this. So, you it is all inviscid flow fluid is moving in mass and then you will find that here the flow there would be if you follow that paper boat you will find that there would be some amount of detour by that path traversed by the paper boat, but then if you put the paper boat here it would there would be hardly any effect away from this.

So, if there is an obstruction there is a log present wooden log present on the flow path then you will have these lines coming in. So, here you will have so these are essentially path lines and streamline though in a very loose way we can call these a streamline, but streamline has a rigorous definition which is given here. Essentially, the streamline and path lines will merge on each other when you have a steady flow. When you have velocity field is steady, then the streamlines and path lines will be falling. One will be falling on the other. Streamline a definition is more rigorous the definition is something like this lines drawn in the flow field such that at a given instant they are tangent to the direction of the flow. So, what that means, suppose this is the flow path, flow path followed this is the line followed by the fluid particles.

So, you can take so, this line you will call a streamline if the tangent to this line gives you the velocity of the fluid particle at that point. Similarly, here it is the tangent to this line would be given will give you the tangent to this line give you the direction of the velocity of fluid particles at that point. So, that is the idea. So, what that means is if fluid particles are moving, the velocity direction is tangential to this line. One immediate implication of this is no velocity would exist perpendicular to the streamline no because if the velocity of the velocity particles is given by the tangent to this line. So, normal to this line, there would be no velocity. It cannot exist.

So, that is also one immediate consequence. So, when you draw streamline then you know for sure that nothing will cross the streamline the flow will be enclosed in that streamline. Now, if that is so, then if you take an element of this line and you call this dr an element of this line which you call dr and if you say that both will have the same direction the velocity of fluid particles at that point and the linear element that you picked up from that line which is dr if both of them are going to have the same direction then one condition one has to satisfy that is $\vec{V} \times d\vec{r} = 0$.

So, now typically we talk about it because three dimensional it is difficult for us to comprehend. So, generally, these streamlines are defined in a two-dimensional manner.

There are certain concepts called stream tubes where streamlines are bundled, and one can see some effect in the third dimension, but with that, we are in this course, we are only talking about two-dimensional flow.

So, all these flows are considering unit depth perpendicular to the screen and unit depth perpendicular to the screen there is no change. So, that is that is tacitly what we have assumed. So, you have the velocity which is it has it has this velocity is we are writing it as $\hat{i}\,u + \hat{j}\,v$. u v are the x and y component and then you have you have written it velocity is u i hat plus v j hat and dr, dr can be also broken down into $\hat{i}\,dx + \hat{j}\,dy$ because dr is an arbitrary length element arbitrary vector displacement arbitrary vector on that on that line. So, that can be always broken down into dx and dy. So, if you take the cross product you will have i cross i and i cross i is non-existent j cross i and i cross j those two terms we are taking.

$$\vec{V} \times d\vec{r} = 0$$

$$\left(\hat{i}\,u + \hat{j}\,v\right) \times \left(\hat{i}\,dx + \hat{j}\,dy\right) = 0$$

$$\Longrightarrow \hat{k}\left(u\,dy - v\,dx\right) = 0$$

$$\frac{dx}{u} = \frac{dy}{v}$$

So, it is and the product becomes k hat because i cross j i cross j is becoming k hat and j cross i becomes minus k hat. So, this is k hat u dy u dy and v dx. So, u dy and minus v dx when we take the cross product. So, k hat into u dy minus v dx equal to 0. So, this takes me to dx divided by u I mean if you write u dy is equal to v dx.

So, we write this is equal to this u dy is equal to v dx. So, you can write $\frac{dx}{u} = \frac{dy}{v}$ that is exactly what we have written. So, this is the equation that one would follow for the streamline that is what is given. So, what are we trying to do here in this context we can come up with something called. So, this is the equation of a streamline dx by u is equal to dy by v.

Now, each streamline will be equal to a unique value of stream function and what that is the way I mean see that the way this stream function was conceptualized that in stream function will make sure that you do not need u and v separately we can simply operate on the stream function and come up with the value of u and v. So, the ψ which is stream function is defined as this can replace two velocity components I do not need two velocity components I just need one function that is ψ. And one thing is for certain that one has to satisfy whatever ψ whatever way you define ψ, but one has to satisfy the continuity equation because that is the that is something we have to follow we have to obey. So, if $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ has to be satisfied we see that if $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$ then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial x\, \partial y} = 0$$

So, if we define psi as this then this is at least this satisfies the continuity equation. Fact is I mean why are there may be a question that why do I get to this stream function this will be clear this will be you will get it clear in a moment because it will be clear from next few slides you will see here that I am trying to come up with some function which will replace u and v number 1 and this function will relate to these streamlines. So, what do I have with regard to streamlines and this is fine this is the definition of stream function. So, here, dr is an element of length along a streamline, and v and dr have the same direction. This is the condition that we have done, and then we have u dy minus v dx equal to 0. This is what we have. So, now, if we put this there instead of u if we put instead of u if we put $\frac{\partial \psi}{\partial y}$.

So, it would be del psi del y dy and minus del psi del x dx. So, already there is a minus sign how did we get u dy minus v dx equal to 0 from there and then now after definition after defining this stream function I am replacing u with del psi del u with del psi del y u with del psi del y and minus v with del psi del x because v is minus del psi del x. So, minus v would be plus del psi del x. So, that is exactly what I have done. So, v is replaced by del psi minus v is replaced by del psi del x multiplied by dx.

So,

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

So, if we do this then what is $\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$ that is essentially d ψ and if this is equal to 0 as per the definition of streamline. So, what this says is d ψ is equal to 0 if we define ψ this way and if streamlines are defined this way. So, d ψ is equal to 0 means psi is constant along a streamline. That means, when I draw these lines this has a particular value of ψ we will get into the unit of that, but psi would be let us say some number I can write it for the time being psi 1 some number would be there 1 2 3 psi is equal to psi 2.

So, all these lines will have a unique value of stream function. So, we do not need to do we know to we do not have to float paper board to get these lines by looking at the velocity field itself we can immediately plot this and then we extract further information where the velocity is increasing where the flow is accelerating where the flow is taking a turn. So, all these details we can get we can we can we can plot streamline by looking at the value of stream function we do not have to flow paper board, but we will from the mathematical expression itself we can generate these curves. And so, that gives me a visualization effect I mean that visual will have a better impact on us rather than those you know algebraic expressions. So, that is the purpose and the genius of this coming up with this stream function psi and linking that to the streamline you must be appreciating I mean the scientists who have come up with these whole concepts it will be clear what would be the what is the utility of this stream function and streamline as I proceed further. So, I will have a formal lecture on all these there would be other functions stream function and all these.

So, next lecture I will start with stream function and potential function under that title I will be discussing. So, now, you have some idea that where we are heading to with these in viscid flow that is all as far as this lecture module is concerned. Thank you for your attention.