Momentum Transfer in Fluids
Prof. Somenath Ganguly
Department of Chemical Engineering
IIT Kharagpur
Week-05
Lecture-22

Welcome to this lecture on Momentum Transfer in Fluids. We were discussing about the motion of fluid particles and in particular we were discussing about the rotation, the condition of irrotationality and how circulation comes in between and saves us for the time being. So, what we had what we discussed at the end of the last class was that circulation it is it has a unique definition and if the circulation is non-zero for a rotational flow as well as for an irrotational flow that means, where the vorticity is 0 still we can calculate a circulation and I can tell the customer I can tell the person professional who is working who wants to know what is the strength of certain vortex I can at least give him some characterization number that the strength of the vortex is this much even if it is irrotational, even if the moment is merely in circular lines. So, in case of for rigid body motion what we have seen is that $v_\theta = \omega r$ that is we said that if there is a movement in a circular path if this is the center. So, then there would be a v, which is radially outward, and there would be a vθ, which would be in this direction perpendicular to this radial direction.

CIRCULATION (*Γ*)

The line integral of the tangential velocity component about any closed curve fixed in the flow,

$$\Gamma = \oint_c \vec{V}.d\vec{s}$$

$d\vec{s}$ is an elemental vector tangent to the curve and having length ds of the element of arc; a positive sense corresponds to a counter clockwise path of integration around the curve.

Relation with vorticity

For the closed curve oacb,

$$\Delta\Gamma = u\Delta x + \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta y - \left(u + \frac{\partial u}{\partial y}\Delta y\right)\Delta x - v\Delta y$$

$$\Delta\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x\Delta y$$

$$\Delta\Gamma = 2\omega_z\Delta x\Delta y$$

Then,

$$\Gamma = \oint_c \vec{V}\cdot d\vec{s} = \int_c 2\omega_z\, dA \qquad \Rightarrow \Gamma = \int_c \left(\nabla \times \vec{V}\right)_z dA$$

Circular streamline $v_r = 0, v_\theta = f(r)$
For a rigid body motion $v_\theta = \omega r$

$$\omega_z = \frac{1}{2}\cdot\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)$$

$$\Rightarrow \omega_z = \frac{1}{2r}(2\omega r)$$

$$\Rightarrow \omega_z = \omega$$

$$\Gamma = \oint_c \vec{V}\cdot d\vec{s} = \int_c 2\omega_z\, dA$$

$$\Rightarrow \Gamma = 2\omega A$$

For irrotational flow $\omega_z = \frac{1}{2}\cdot\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = 0$

$$\Rightarrow r v_\theta = constant \Rightarrow v_\theta = \frac{c}{r}$$

$$\Gamma = \oint_c \vec{V}\cdot d\vec{s} = \oint_c^{2\pi} \frac{c}{r}\cdot r d\theta = 2\pi c$$

Where A is the area enclosed by the contour

So, this we call v_θ. So, we one thing is for sure that a vr would be equal to 0, v r would be 0 in whether it is e-rotational or it there is viscosity involved, but v theta for a rigid body motion it would be ω into r this is something which we have studied in engineering mechanics. So, if you put v theta as ωr and then we try to find out what is the ω z. Now, ωz though we have talked about it in terms of only the Cartesian system if we talked about the curl of the velocity field in for Cartesian system curl of the velocity field for the Cartesian system.

Similarly, I will have another equation for cylindrical system just like we had earlier grad of p in cylindrical system there were or grad of v there were in cylindrical system there is similar equation, but there are other terms coming in same exercise one has to do with ω z and they can they may see that ω z if you try to write it in terms of r theta coordinate instead of Cartesian x y coordinate it would be $\omega_z = \frac{1}{2} \cdot \frac{1}{r} \frac{\partial}{\partial r}(rv_\theta)$ that is how it is. So, you can I mean if you are willing you can find out what would be the definition of curl in cylindrical coordinate system. So, all the components ω x ω y ω z in cylindrical coordinate system will be available So, if you apply this if you put $v_\theta = \omega r$ and take the derivative with respect to r because now it becomes r square ω derivative of r square ω would be 2 ω r and outside there is 2 r. So, ω z becomes equal to ω. So, if there is a rigid body motion v theta is equal to ω r then the z becomes ω.

Ω z is basically the rotation in x y plane. It has a rotate for a rigid body motion it has a angular velocity ω z which is ω. So, now if you want to do this now if you if you want to utilize this then you will write for the full circle anti-clockwise typically we do this we put an arrow here that to show that this circle we have to count it as anti-clockwise you have to go then circulation will be positive. So, for this anti-clockwise v.ds. So, this would be equal to we call it $2\omega_z dA$.

So, we take it here as $2\omega_z dA$ and then if we for a finite value of A. So, if this if this circulation is happening over an area A. So, this integration will take you to 2ωA because ω_z is equal to ω already we have seen. So, circulation has a value which is equal to 2ωA for a rigid body motion circulation has a value when the vorticity is non-zero. So, for rotational flow for that means, when the viscous core is present within the viscous core you can have a circulation which is given by2ωA.

When it comes to irrotational flow we said that vorticity is 0, but irrotational flow if we try to find out what is ω_z the ω_z would be

$$\omega_z = \frac{1}{2} \cdot \frac{1}{r} \frac{\partial}{\partial r}(rv_\theta) = 0$$

Same equation we are putting here and ω_z has to be equal to 0 because that is what the condition is flow is the differential element is not rotating around its own axis. So, in that case if this is 0 then del del r of r v theta equal to 0. So, that means r v theta would be equal to constant because they are only a function of r. So, now, it is treated as if the derivative with respect to r is 0.

So, rv_θ would be equal to constant and this is so, this gives me the constant value of the constant is C then v theta would be equal to C by r. So, we have now the gamma would be

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s}$$

and v is in this case v is v_θ. So, v_θ is c/r and you have let us say as far as this particle that is moving in a circular path it has a velocity perpendicular there is no velocity in radial direction perpendicular to this it has a velocity that velocity is v_θ and let us say it makes an angle. So, this is let us say theta this angle is theta this angle is theta and this is the differential angle d theta. So, over this d theta over this d theta small d theta the velocity will remain same as v theta because if we look at here the theta changes.

So, v theta changes, but over this small arc length I can assume v theta to remains to remain same as we have done for other cases. So, this v theta in that case we write $v_\theta = \frac{c}{r}$ as is given here and the arc length here would be equal to $rd\theta$ because this radius is r this radius is r this radius is r and this angle is d theta this small angle is d theta. So, this length this arc length this arc length would be $rd\theta$. So, if this arc length is $rd\theta$. So, we multiply that arc length C by r is v theta v theta into r d theta and now we do this integration over theta changing from 0 to full circle 2π.

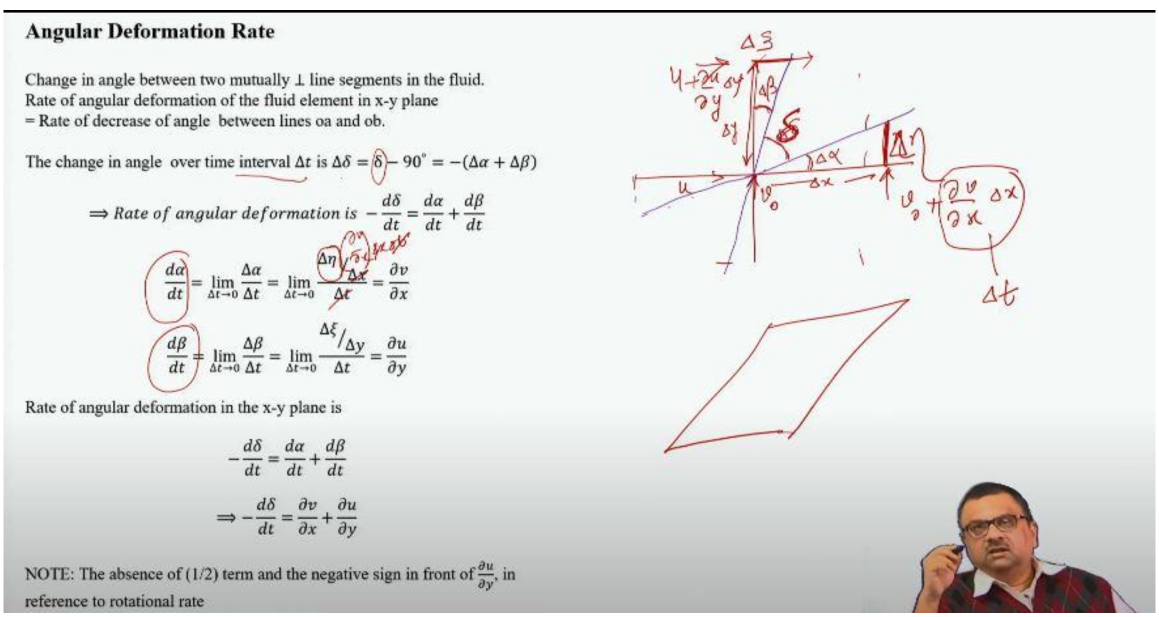
So, theta changing from 0 to 2 pi C by r r d theta. So, if we assume that I am not talking about r the case where r is equal to 0. In fact, we cannot talk about that either because I said at the center of the vortex there will be a viscous core existing. So, as long as we are not talking as long as we know r is greater than 0 r is not equal to 0 we can simply cancel r and we are left with C d theta integral C goes outside. So, you get $2\pi c$.

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s} = \oint_c^{2\pi} \frac{c}{r} \cdot rd\theta = 2\pi c$$

So, that is the circulation. So, now, you have a case where you straight away assumed ω z to be 0 the differential element is not rotating around its axis, but still you have a circulation you can tell others the strength of the vortex is $2\pi c$. So, this is the advantage of having another parallel way of expressing the movement in circles other than vorticity. So, this we have already discussed that rotation of fluid element about z, x axis and y axis and we have mentioned that this is now for a Cartesian system $\vec{\omega} = \frac{1}{2}(\nabla \times \vec{V})$, that is this curl of the velocity field just for a cylindrical coordinate system there would be a definition as well. So, that definition you can find out from the references cited.

Now, so far we have talked about what all we talked about the translation, we had talked about the acceleration and substantial derivative, we talked about the rotation and now we are going to talk about something called angular deformation. Once again we from our knowledge of the definition of viscosity we know that the viscosity arises that Newton's law of viscosity itself says that shear stress is proportional to rate of angular deformation. So, when it comes to angular deformation it is totally in the domain of viscous flow, but we need to understand what viscous flow means. For example, I give you an some arbitrary velocity field and I will ask you can you tell me what whether it is whether there is a rotation involved whether you call it inviscid or you call it viscous. So, at least you can quickly run a curl of that velocity field and see whether it is 0 or non 0, and from there, you can identify whether we are into the inviscid or viscous domain.

So, similarly the angular deformation also one can check one needs to check because angular deformation arises once again because of viscous flow. So, what is this angular deformation here? It is very similar to what you have for rotation. So, rotation was like this. So, rotation you had this one moving and that one also moving in the other direction. In this case, the movement would be slightly somewhat different.



The movement would be this line would be moving this line the modified line would be this would be the modified line let us say this line is taking this shape. The purple line is the new line that is created and this vertical line this is taking a shape like this then you call it the angular deformation not the rotation. Rotation would have moved in the other direction, but that has not taken place. Obviously, any flow will have a mixture of rotation and angular deformation. So, if I give you a velocity field immediately you can find out what is the extent of angular deformation in that and what is the extent of rotation in that.

So, both would be present simultaneously to some extent rotation some extent angular deformation. If we are working only with angular deformation you would have expected this type of a plot. So, you can see that now, if you had drawn this line, this box is as if this was the box originally, and now the revised box is something else. The revised box took shape. Now, this is the central line that is the central line. So, you can see the extent of deformation.

So, I can probably make an attempt to draw this it would be something like. So, this would be one line and this would be the other line. So, it is not rotation. It is taking a shape like this. It is shape also changing. So, in that case, if the central lines are moved that way, then you will have you will see a deformation, something of this sort. So, let us say I write these as Δ α and these as Δ β now it is in other direction positive x direction and then we try to apply the same concept that we had used for rotation and try to find out what is the $\frac{d\alpha}{dt}$ and what is the $\frac{d\beta}{dt}$.

Now, mind it if this angle between the two is Δδ. So, total it has to be Δ if this total let us say Δ plus Δ α plus Δ β is equal to 90 degree. So, this angle must be Δ in that case. So, Δ α plus Δ plus Δ β this is the sum of these three angles will be 90 degree, but originally your Δ was 90 degree because these two lines were mutually perpendicular now the lines have deformed. So, what is the change in Δ over time interval Δ t that is Δ present value minus the initial value was this

$$\Delta\delta = \delta - 90^{\circ} = -(\Delta\alpha + \Delta\beta)$$

So, rate of angular deformation if you call it then this would be minus of d Δ dt if I take d dt of these 90 degree derivative of time is 0. So, it is minus d Δ dt if I take the minus sign on this side. So, minus d Δ dt would be equal to d α dt plus d β dt. So, now, we need to find out what is d α dt and what is d β dt.

So, d α dt is limit Δ t tending to 0 Δ α by Δ t again that same Δ α would be what Δ α would be that same this was this we said it is eta and this we said it was zeta right Δ eta sorry we talk about Δ eta and we talked about Δ zeta. So, we would be and we would be saying Δ α if it is small then Δ α can be replaced by tan Δ α and tan Δ α would be Δ eta by Δ x. So, Δ α is replaced by Δ eta by Δ x and these divided by Δ t limit Δ t tending to 0 and the same logic that we have put for rotation in the last class in the last lecture module you can see that this would be what was our consideration here the velocity is Vo here the velocity is Vo plus del v del x Δ x because this distance is Δ x and we said that this is the extra velocity that this point has and so, this extra velocity when I multiply it with Δ t for Δ t time this point experience the extra velocity and because of that this point has moved by an extra amount Δ eta. So, Δ eta is simply del v del x Δ x Δ t. So, the same logic Δ eta would be replaced by del v del x Δ x Δ t and del v del x Δ x Δ t and these Δ x and Δ t will cancel with this Δ x Δ t.

$$\frac{d\alpha}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\eta / \Delta x}{\Delta t} = \frac{\partial v}{\partial x}$$

$$\frac{d\beta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\beta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\xi / \Delta y}{\Delta t} = \frac{\partial u}{\partial y}$$

So, you are left with only del v del x. Similarly, d β dt when you do d β dt here here we exactly do the same exercise the only change here you can see is that the Δ zeta is also in positive x direction and the velocity which happened to be u here and it is u at this point and it is u plus del u del y Δ y because this distance is Δ y. So, same that Taylor series expansion and working with only first order term. So, this is u plus del u Δ y. So, this velocity so, this is the extra velocity by which this point is moving.

So, we multiplied it by Δt. So, over Δt time there would be some extra movement and that movement is equal to zeta. Only point here is last time when we worked with there the movement was in the negative direction. So, that is why we had a minus sign, but here in this case there is no minus sign. So, it would earlier it was a minus del u del y and you may recall ω oa we have written as half of del v del x minus del u del y, but here we do not have that minus sign we have del v del x and del u del y and if we since minus d Δ dt is sum of these two. So, we have to simply write it as del v del x plus del u del y.

So, one must note the absence of that half term here we are not averaging we are following this equation. So, half is not there and the negative sign is not there negative sign was there in case of rotation. So, this is as far as the rate of angular deformation in the xy plane. So, similarly you will have rate of angular deformation for. So, this is the xy plane del v del x plus del u del y.

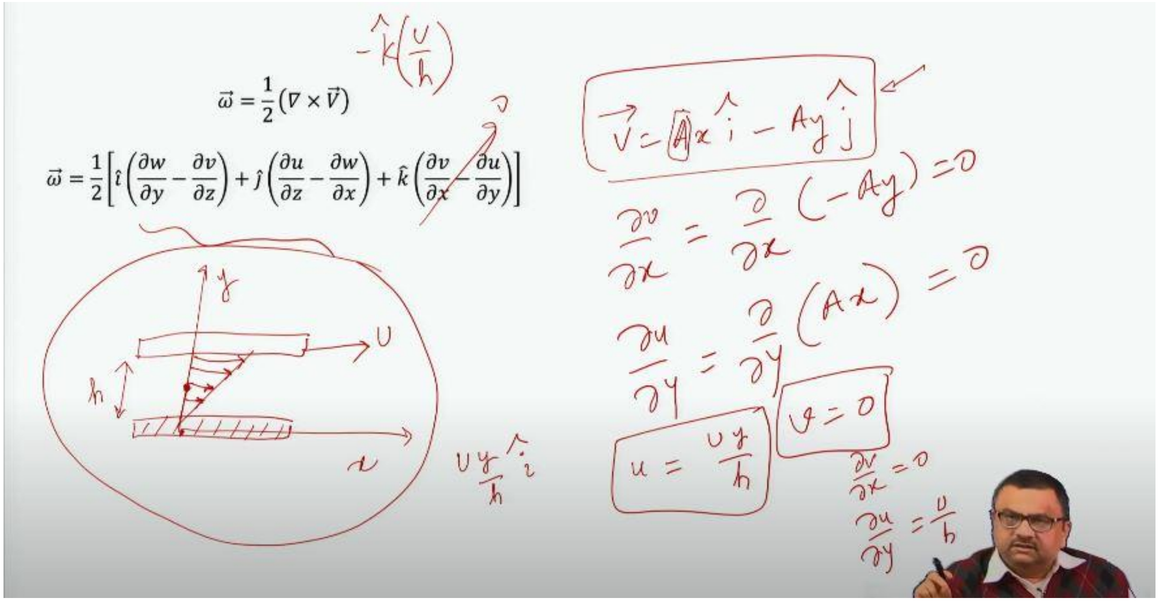
Similarly, you will have angular deformation in yz plane and similarly you will have angular deformation in zx plane. So, these values will come. So, the difference here is that we are not having that half and that minus sign is not there. So, that is how angular deformation is defined. So, what we will what we expect here is if I have some velocity field I give you and then you first of all you find out whether there is any rotation involved whether there is any angular deformation involved.

So, when you see these are not there then you can consider rotation is not there you know it is the flow is irrotational. Angular deformation you can check and then you can proceed with the inviscid flow treatment, but if these components are nonzero then you have to bring in you know that there is a viscous flow happening. So, what is the I mean how do I know which one is the I mean by looking at the velocity field. So, let us look at the velocity fields so, far we have worked with we have worked with one velocity field you may recall where we said that to find continuity if u is given as Ax what could be a possible v such that this continuity is valid. So, u is equal to Ax and we found there it is I think two lectures before this I mean earlier and we found there that v is equal to minus Ay.

So, that means, the velocity field in that case is Ax i hat minus Ay j hat. So, this is one possible combination where the continuity is satisfied. Now, you can see quickly whether this velocity field is rotational or irrotational. So, these velocity field is you will see that what was the condition of rotation first of all this is in x y plane i and j is involved k is not there k hat is not there. So, we do not have w at all anywhere and what was our velocity components for rotation I can see that w in that case is not there w is not there and z there is no variation with z.

So, you will be left with only this expression this part as the ω . So, del v del x minus del u del y because our expression was if I write it again if you allow me Ax i hat minus Ay j hat. So, I can see here what is del v del x and remotely that is a possibility del v del x is del del x of minus Ay and that is equal to 0 del u del y that is equal to del del y of Ax and that is equal to 0. So, you can see here that this term goes to 0 and this term these two terms they do not come into play at all. You can see here in this case the curl of the velocity field is 0.

So, you know that the flow is very much irrotational there is no rotational aspect involved, but then what is the rotational flow? Rotational flow would be the case where you remember our we talk about the viscous we talk about the definition of viscosity lower plate is fixed and upper plate is moving at a velocity at a constant velocity. So, we saw that the velocity profile if we assume that the velocity profile is linear of course, they it needs certain assumptions the this gap between the two plates is small the plates might be infinitely wide and there are other assumptions that would be addressed separately in this course, but these this if you if you if you have a linear velocity profile then if these gap is let us say h and these velocity is capital U and this direction is y and this direction is x. So, velocity at any layer y distance away. So, let us say at this point y distance away from the bottom plate because origin is here.



So, that is how y is counted. So, you will have the velocity there would be typically U into y by h U minus 0 you can write because it velocity is varying from 0 to U it is just a linear interpolation if you do. So, U minus 0 into y by h. So, that gives you the velocity at that location U minus 0 is U. So, this becomes the velocity in which direction i hat in j hat direction there is no velocity and perpendicular to the screen there is no velocity. Now, if you have so, that means, if this is so, then you know that you have only the U component of the velocity present and the well magnitude of that is capital U which is a constant velocity by which the upper plate is pulled into y divided by h.

So, now, if you if you have these as U and V as 0. So, what would be the del V del x and what is the del U del y? Del V del x del V del x is 0 because V is 0, but del U del y is U by h. So, you will have here a value of ω here you have the curl of velocity field existing because del U del y del V del x is 0 I agree, but del U del y that would be equal to U by h derivative of this with respect to y is non 0 it is U by h. So, this becomes equal to k hat into del V del x 0 minus this.

So, this is minus k hat U by h. So, that becomes the half of this. So, that becomes this becomes the curl of the velocity field. So, you have the rotation involved here and you if you if you want to find out the angular deformation you can you can also see that angular deformation was what? It is only that there was no minus sign and half was not there right that is what we said. So, if the half is not there and other terms are not a non existent, but k hat del V del x plus del U del y del V del x is 0 plus, but del U del y is existing it is U by h. So, you have only thing is it was half here you have the half of whatever you get in case of rotation it that half term will not be there, but it would be simply U by h again angular deformation.

So, this mode of flow will very much be rotational here the here the differential element is rotating getting rotated around its own axis it has angular rotation angular deformation it is not only rotated around its axis it gets deformed in a there is angular deformation happening both are existing in this in this in this mode of flow. Whereas, when it comes to in fact, coming to think of it what kind of flow is this probably this is something which you may which you may attempt before we get to the next lecture what type of flow is this you can even put some values of let us say A you put some value 1 x i hat minus 1 it has it has to have a unit because see this is the unit of meter per second x already has a unit of meter. So, A must be having second inverse as the unit. Let us say you take one second inverse as a value of A and try to find out the velocity at various locations and see what kind of velocity are we looking at here I can see the velocity is moment when you have two plates one is fixed and the other plate is moving at a velocity U. So, what would be the what would be the velocity at the intermediate locations, but in this case what would be the velocity.

So, that you may like to look into before we start the next class I will try to also review that quickly. That is all as far as this module is concerned I will in the in the next lecture I will complete this angular deformation and then I will I will look into that circular path that we talked about and we mentioned something about streamline we want to discuss this more rigorously that was a very course way of telling. So, we will we will talk about it in the next lecture module that is all as far as this one is concerned. Thank you for your attention.