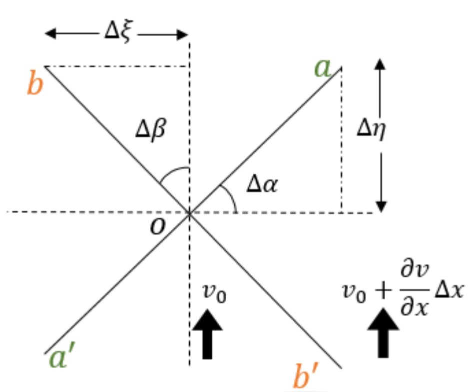
Momentum Transfer in Fluids
Prof. Somenath Ganguly
Department of Chemical Engineering
IIT Kharagpur
Week-05
Lecture-21

Welcome to this lecture on Momentum Transfer in Fluids. We briefly introduced the motion of fluid particles. We have arrived at some force balance equations, and we found out that there are different forms of motions that are possible. One is translation; the other is rotation, the third one is angular deformation, and the fourth one is linear deformation, and any movement of the fluid particles would be a combination of all these. Now, out of these, some are components of inviscid flow, in some cases, the concentration of viscosity is important. For example, I have mentioned rotation and angular deformation, their shear stress plays a role.

So, we have to differentiate what I mean when we cannot consider a velocity field to be inviscid and when we consider the velocity field to be viscous by looking at that velocity field. So, for that reason, we need to understand the various components of these motions of fluid particles. Out of that translation we have already talked about and we said that that is essentially the acceleration and substantial derivative we have discussed. I mean, what are the considerations there due to these assumptions in the Eulerian framework?



The other thing that we were looking into is the rotation, and we said that the rotation, we here we are looking at the rotation of the particle that is in xy plane. So, any rotation over the xy plane that we will call ω_z because the plane has the direction perpendicular to the plane, and that is the z direction, which is, in this case, perpendicular to the screen. So, what we briefly mentioned that how we can find out by looking at the velocity field how I can find out whether that velocity field has any rotational aspect in this. So, for this, we said that I have taken a cross where a prime b b prime, and then we said that hereafter at time t, let us say they are mutually perpendicular to each other, and this is how the cross well looks after time Δt. So, at that is at time $t+\Delta t$ there is a rotation that comes in and so what we see here is that the aa' which was horizontal now it has become slanted, and since the other one will remain in this direction the b b prime would be taking this position.

So, first of all why did this rotation take place had it been that all point o and point a if they were moving at same velocity there was no possibility of a rotation it is the rotation is happening because point a is moving at some extra velocity compared to point o. So, that is why we said that it is. I have taken the let us say the velocity at this point is vo. We are assuming that the velocity at this location we are taking a Taylor series expansion of velocity, and we are only taking the first order term. So, that is why $v_0 + \frac{\partial v}{\partial x}\Delta x$ is this length. So, this length or we can call this as the Δ x.

So, over this Δ x the velocity changes velocity has to be higher at this location compared to point o then only there would be some extra movement coming in with point a. So, what we said is that this point is moving at a velocity v o and this point is moving at a velocity v o plus something and that is from Taylor series expansion. Now, over Δt time if we look at this is the Δt time. So, over Δ t time these being the extra velocity that you have if we multiply this by Δ t. So, that gives me.

So, this term gives me the additional movement the point a will undertake compared to point o and that is essentially $\Delta\eta$. So, that is why we have written $\Delta\eta$ you may you may see this Δ eta is replaced by we can we can replace this Δ eta by $\frac{\partial v}{\partial x}\Delta x$. So, that is that is what Δ eta is. So, now, what we would say is that as far as the line oa is concerned it is having an angular velocity since there is a rotation and if this angle is Δ alpha. So,

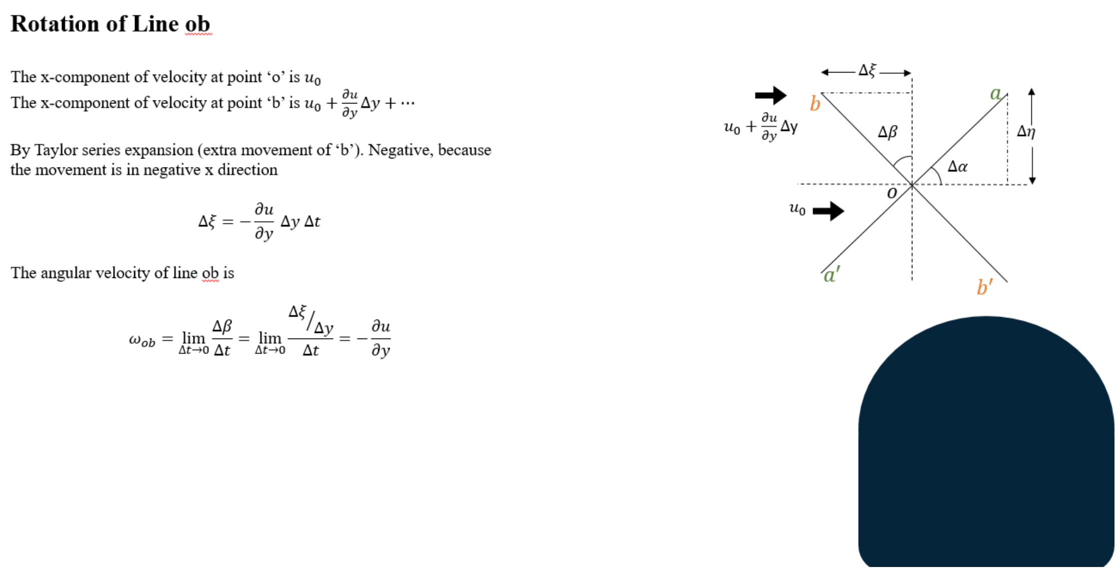
$$\omega_{oa} = \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\eta / \Delta x}{\Delta t}$$

that is the definition of angular velocity and limit Δ t tending to 0 Δ alpha we will write Δ alpha as $\Delta\eta / \Delta x$.

So, what are we how are we simplifying it? We are saying theta is theta and tan theta they are same for small value of theta. So, that is that is why we are writing alpha as $\Delta\eta / \Delta x$ and then this $\frac{\Delta\eta / \Delta x}{\Delta t}$ and this Δ eta is that extra movement it has. So, that we put as del v del x Δ x. So, this Δ x and this Δ x will cancel out. So, what you have here is del v del x divided by Δ t.

So, Δ eta is the extra movement. Wait a second here I think it is this is there is a small issue Δ alpha by Δ t and that is that is ok I have one Δ t term here that I missed out del v del x Δ x Δ t. So, del v del x Δ t. So, this Δ x will cancel with this Δ x this Δ t will cancel this Δ t. So, essentially you get $\omega_{oa} = \frac{\partial v}{\partial x}$.

So, that is as far as the rotation of oa line is concerned. What will happen to ob? You may say you may argue that if this is if they have to remain perpendicular to each other then this Δ alpha has to be equal to Δβ. So, Δ eta has to be equal to Δ zeta something of that something in that line. So, if we keep an open mind and treat these as Δβ let us see where we go. So, for ω_{ob} what we see here is if we focus on this rotation of line ob.



So, we see here that this angle it is Δ beta and this point is traveling at a velocity u_o. Last time we had v o because the velocity is in y direction this is the y direction this is the x direction. So, this is the v_o since the velocity is in y direction this is $v_0 + \frac{\partial v}{\partial x}\Delta x$ that is the what Δ x the velocity is changing velocity is varying that is why the rotation takes place. But in this case it is the velocity in x direction that causes rotation. So, this is the u at o instead of vo moving upward now we have uo moving going horizontally.

And here at this point the velocity would be $u_0 + \frac{\partial u}{\partial y}\Delta y$ because this distance is Δy this distance is Δ y. So, this one is moving at some velocity this one is moving at some other velocity that is why there is some extra movement here and that we will multiply by Δt which would be the extra movement which we will link to Δξ that is the idea just the way we have done for Δ eta. So, now, here in this case you can see that

$$\Delta\xi = -\frac{\partial u}{\partial y}\,\Delta y\,\Delta t$$

the only issue there is that there has to be a minus sign here. This minus sign arises because here, the movement is taking place in the negative x direction, whereas the velocity is in the positive x direction, which was not the case in the case of oa. So, oa both are positive.

So, you could work with the positive del v del x, but here there would be a minus sign. Otherwise that it is one and the same thing $\Delta\xi = -\frac{\partial u}{\partial y}\,\Delta y\,\Delta t$ that same treatment

$$\omega_{ob} = \lim_{\Delta t \to 0} \frac{\Delta\beta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\xi/\Delta y}{\Delta t} = -\frac{\partial u}{\partial y}$$

And then these Δ zeta is replaced by this quantity and then this Δ y Δ t will cancel with the Δ y Δ t. So, you are left with minus del u del y. So, now, we have two rotational I would say two angular velocity one is ω oa another is ω ob as far as the plane xy plane is concerned.

So, I have a choice here basically these are linking link to ω z ω z k hat I have this $\vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$ and here in this case we have this is linked to $\omega_z\hat{k}$, but which one is linked should we go by ω oa or should we go by ω ob one may say that both are one and the same we do not want to because the problem is that one in one case it is minus del u del y in the another case it is del v del x. So, I mean it is so, will you be will you be defining rotation in terms of only one component of the velocity or will you be give a preference to both. The rational thinking here would be take ω z as half of that is average of ω oA plus ω ob. So, that both I take care of del u del y as well as del v del x. I am not basing my analysis only to the v component, I am not basing my analysis only to the u component.

I am giving equal opportunity to both these or equal importance to both these velocity components. So, that is how the ω_z is defined. So, that is why you have you can see here that

Rotation of fluid element about z-axis = Average angular velocity of two mutually ⊥ line elements oa and ob in x-y plane

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Similarly ω_x = rotation rate of pairs of ⊥ line segments in y-z plane

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\vec{\omega} = \frac{1}{2}\left(\nabla \times \vec{V}\right) = \frac{1}{2}\mathrm{Curl}\,\vec{V} = \frac{1}{2}\vec{\xi} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The fact is I mean the best way to remember this is we need to this is referred as curl of the velocity field.

This should be del v del z. There is a mistake, I believe, and this would be this would be del u del z minus del w del x. This is just fine. So, this is should be del v del z. So, this is ω is half, but the best way to remember this is linking this ω to this is called curl of the velocity field. So, you have a velocity field, velocity field is u i hat plus v j hat plus w k hat. So, what would be this ω in that case? So, ω would be half of half outside and you can write it in the form of a determinant half i hat del w del y minus del v del z.

So, i hat goes with ω x. So, ω x is half outside del w del y minus del v del z. Similarly, with j hat the ω y is half del z I mean if I write additionally this i and j and here I write del del x u and del del y v. So, when I write when I take j when I take j. So, it would be del u del z minus del w del x del u del z minus del w del x that is exactly what we see here and when it comes to k hat which is the ω z.

So, ω z for ω z it would be del v del x minus del u del y del v del x minus del u del y. So, it is easy to remember ω in the form of determinant type form you may remember and this is essentially referred as curl of the velocity field which is which is referred as del cross v. So, curl of the velocity field. Now, what we note here is that the ω which is the angular velocity that is equal to half of whatever velocity field you have you find out what is the curl of this velocity field, and then you calculate it you will get the angular rotation and in i hat j hat and k hat components of that. This curl of this velocity field this is there is also another name attached to it; it is like a zeta vector, which is referred to as vorticity.

So, vorticity of the flow. So, you the velocity field when you see that del cross v is equal to 0 then you will call this flow to be irrotational when del cross v is equal to 0 and when del cross v is nonzero then there is a rotation involved and rotation involved means it is a viscous flow shear stress is involved. So, those issues will come up. So, inviscid flow when we talk about one of the condition is that the del cross v is equal to 0 which is irrotational flow. Now, the fact is a flow can be irrotational in part, and in some other part, flow can be rotational. I can give you quickly an example think of a vortex suppose there is a vortex going in here.

So, a vortex is going in here a swirl so to say in layman's term and you have a small straw that defines your differential element. So, suppose I put this differential element a small straw. So, I want to see how this differential element rotates. So, I may see if I am far away from the center of the vortex, I may see that this differential element simply follow a circular path, but within this different fthis differential element itself is not rotating like this. But rotation when it comes to rotation ω_{oa} ω_{ob} that requires that the differential element has to rotate that is exactly what we done here.

ω_{oa} this was the cross wear this was the cross wear and the cross wear is rotating. So, if this I considered as a differential element there could very well be possible it could very well be possible that the differential element simply is rotating which is simply traveling a circular path without rotating around itself. So, then the flow is irrotational, but as you come close to the center of the vortex you will find very close to the vortex you will find that this differential element will start rotating around its own axis like this. So, in that case that would be the rotation. So, for the major part, it is the irrotational flow, but for some, you have to define some threshold radius.

So, some threshold radius has to be defined beyond which it would be irrotational even though it is moving in a circular line, but it is irrotational, but less when the radius is less than that threshold value the differential element itself is rotating around its own earth axis and that becomes the flow becomes rotational. So, there is viscosity involved. So, generally all vortex will have something called a viscous core and inviscid periphery. So, at the core at the variable within a few millimeters near the center of this vortex, it would be viscous. They are calling it viscous core. So, there the irrotationality is no longer valid.

So, inviscid flow assumptions had to be changed shear stress has to be considered there. So, this is how we this rotation plays out rotation or irrotational flow etcetera these play out. So, I mentioned here the development of rotation requires shear stress on the surface. It cannot develop under the action of body force, that is, gravity, or normal surface force, that is, pressure. So, that is why the equations that we had worked with earlier those were there we have not considered shear stress. So, that is why we said those are applicable for inviscid flow.

So, but we must be we must be aware of this assumption that this there is a condition of rotationality. Now, there is a slight problem. Problem in this sense this is the definition of ω

$$\vec{\omega} = \frac{1}{2}\left(\nabla \times \vec{V}\right)$$

and this is how we have defined ω and we need to look into the angular deformation. But before that, I have to point out one small thing here that is when we say that for the same swirl, the same vortex outer part is irrotational, the inner part is rotational, then I would say that if this is the rotationality condition. So, for the inner part there will be a vorticity finite vorticity, finite ω values existing outside this there is vorticity is 0.

So, then the question would be that there is still some rotation going on. I mean, even if the differential element is not rotating around its own axis, it is still following a circular line. So, there is some rotation happening. So, how to how to so, that the fact of the matter is vorticity or ω, ω is not the not a complete term to explain the as far as the rotational movement is concerned. This only accounts for the rotation of a differential element around its own axis.

CIRCULATION (*Γ*)

The line integral of the tangential velocity component about any closed curve fixed in the flow,

$$\Gamma = \oint_c \vec{V}.d\vec{s}$$

$d\vec{s}$ is an elemental vector tangent to the curve and having length ds of the element of arc; a positive sense corresponds to a counterclockwise path of integration around the curve.

Relation with vorticity

For the closed curve oacb,

$$\Delta\Gamma = u\Delta x + \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta y - \left(u + \frac{\partial u}{\partial y}\Delta y\right)\Delta x - v\Delta y$$

$$\Delta\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x\Delta y$$

$$\Delta\Gamma = 2\omega_z\Delta x\Delta y$$

Then,

$$\Gamma = \oint_c \vec{V}\cdot d\vec{s} = \int_c 2\omega_z\,dA \qquad \Rightarrow \Gamma = \int_c (\nabla\times\vec{V})_z\,dA$$

Circular streamline $v_r = 0, v_\theta = f(r)$
For a rigid body motion $v_\theta = \omega r$

$$\omega_z = \frac{1}{2}\cdot\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)$$

$$\Rightarrow \omega_z = \frac{1}{2r}(2\omega r)$$

$$\Rightarrow \omega_z = \omega$$

$$\Gamma = \oint_c \vec{V}\cdot d\vec{s} = \int_c 2\omega_z\,dA$$

$$\Rightarrow \Gamma = 2\omega A$$

For irrotational flow $\omega_z = \frac{1}{2}\cdot\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) = 0$

$$\Rightarrow rv_\theta = constant \Rightarrow v_\theta = \frac{c}{r}$$

$$\Gamma = \oint_c \vec{V}\cdot d\vec{s} = \oint_c^{2\pi} \frac{c}{r}\cdot r d\theta = 2\pi c$$

Where A is the area enclosed by the contour

So, if we want to have a more general term. So, for this reason, there is a new term which is used in this context, which is which goes by the name circulation. Circulation is defined as

$$\Gamma = \oint_c \vec{V}.d\vec{s}$$

Line integral of the tangential velocity component about any closed curve fixed in the flow. That means, if you if you have this is your x axis, this is your y axis. So, you will have a closed curve would be oacb.

So, you will go here, you will go there, you will go there and then you come back. So, if you if you want to do that. So, this is let us say here the velocity is we already said this is u, here the velocity is we already said it is v, this side is Δx, this side is Δy and here the velocity is here the velocity is we have already mentioned it. It is if this is v then it is $v + \frac{\partial v}{\partial x}\Delta x$ right ignoring the higher order term. We have already seen this in the in the context of rotation.

Similarly, here you will have a velocity which is which is here it is u then here it is if this is u then then here it would be $u + \frac{\partial u}{\partial y}\Delta y$ and here it is if it when it comes backward in this direction. So, so in in this direction this would be the this this with a negative sign would be the velocity and in this direction it would be v Δy that would be the that that would be the velocity just like u Δ x it would be the. So, so what I am trying to do here? I am trying to traverse a path tangential velocity component about any closed curve. So, this is my closed curve and within this closed curve I am trying to find out what is the what would be the v.ds. So, what is the v here? It is u if I follow this curve u into Δ s v dot ds, ds is the displacement.

So, ds is the elemental vector tangent to the curve and having length ds of the element of arc a positive sense corresponds to a counterclockwise path of integration around the curve. So, here I have these as Δ x. So, u Δ x that takes care of v dot ds as far as this path is concerned then I have to follow this path which is v plus del v del x Δ x into then this has to be multiplied by Δ y because this distance is Δ y. So, I take care of v dot ds as far as this path is concerned then I come c v, but here I am going negative x direction. So, there is that is why I have a minus sign and here I have u.

So, here I have the velocity u plus del u del y Δ y. So, that is why this multiplied by Δ x because this distance is Δx. So, I complete the path up to this point and then I have to travel again downward that means, in negative y axis at a velocity v Δ y. So, once we do this we find that if we these terms will cancel out for example, v dot Δ y will cancel with v dot Δ y u Δ x will cancel with u Δ x. So, you end up with Δ gamma as far as the differential path is concerned Δ gamma is essentially

$$\Delta\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x \Delta y$$

$$\Delta\Gamma = 2\omega_z \Delta x \Delta y$$

Now, $\frac{\partial v}{\partial x}$ is $2\omega_z$ because ω z was half del v del x minus del u del y. So, Δ gamma is then defined as 2 ω z Δ x Δ y. So, now, if you want to do the circulation this is for a differential element then you do the integration v dot ds over the path over that entire closed curve and then it would be simply 2 ω z Δ x Δ y is the area over which this ΔA is the differential area enclosed by this contour. So, that is why you write this gamma as closed path C over the entire closed curve curl of the velocity field the z component of that and multiplied by dA dA arising from Δ x Δ y.

So, this is what is gamma. So, if you have a rotation in the classical sense just the way we have talked about OA, OB and having rotation. So, you can see that if there is a rotation involved circulation is non-zero because this curl of the velocity field in the z component that would be non-zero. So, vorticity is non-zero circulation also has a value. So, circulation is non-zero when it comes to regular rotation regular rotational flow because of shear stress and other things. On the contrary, when you have a movement along a circular path, we call this streamline. I have not defined streamline as yet, which I will do either probably in the next lecture.

So, that is the path that I am talking about for the time being if you consider this because it has a more rigorous definition it is not just the path, but it is if something is traveling along a circular path. So, vr is typically 0 when something is rotating like this and if this is the center. So, this is considered r and this is considered this angle is considered theta. So, you will have if the rotation is if the movement of the particle is in this circular path then you will not have any particle moving perpendicular to the path. So, v_r will be invariably 0 and v theta which is the direction in this direction there would be that v theta would be existing.

So, this is something which we will call which we will have for a flow in a circular path. Now, here I have two issues here one is called a rigid body motion another is called irrotational flow. So, for irrotational flow what we are trying to show here I mean what we will do probably in the next module in the next lecture module is that we will try to show that for a rigid body motion you it is irrotational sorry for a rigid body motion it is a rotational flow there you have non-zero circulation and for irrotational flow also it is the circulation is non-zero. So, that means, circulation happened to be a more general term because see we when there is a movement in a circular path whether it is a viscous core or in viscid peripheral movement one needs to know what is the strength of that I mean this is a vortex what is the strength of this vortex is it too strong a vortex then you when you are doing an engineering design then you have to you have to keep that in mind what. So, if someone wants to know that tell me what is the strength of the vortex how protective I should be against that vortex.

So, then if you say I have a vortex, but then what is the strength you say strength I give by body city and that is 0 that nobody will accept I mean I mean the people who are in the who are implementing it. So, that is why this circulation will come in I will continue this in the next lecture and I will show the difference between circulation and the rigid body motion and the irrotational flow circulation will have both non-zero value for both. So, you can characterize a vortex to a better extent, that is all as far as this lecture module is concerned. Thank you for your attention.