Momentum Transfer in Fluids
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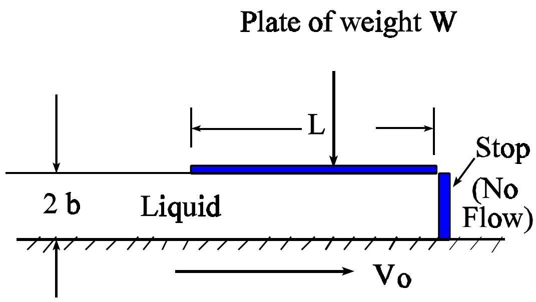
Week-04
Lecture-20

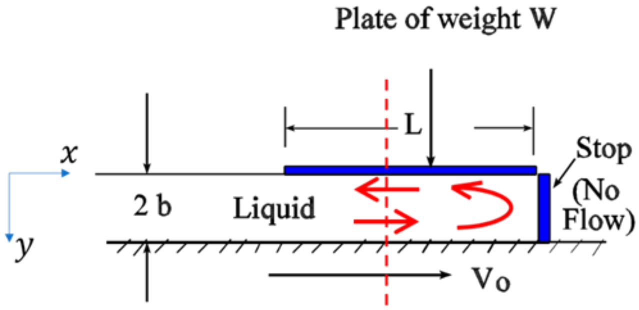
This is going to be our next treatment of the use of Navier Stokes equation. So, what I am trying to show to you is that the, it is easy to use Navier Stokes equation, but it is equally important that we understand the physics of it. Because the physics of the problem is not only expressed in terms of the governing equation, but more importantly in terms of the boundary conditions. At some point it could be also necessary to identify even if it is not specifically mentioned that what could or could not be present in a specific situation. So, the next problem that we are going to deal with will give us some idea of what is to be, what is to be understood, what kind of a what kind of force or forces would be present to satisfy the condition that is stated in the problem. So, if we look at the problem statement here, what we can see is that we have a lubricated thrust bearing which is moving at a velocity.

So, we are going to continue with our treatment of Navier Stokes equation and you would see that it is not only necessary to simplify the Navier Stokes equation to obtain the governing equation. At some point of time, you also must think the statement of the problem and how that can be converted or how that would give rise to additional terms in Navier Stokes equation which are not apparent. And secondly, the physics of the problem is also expressed in an accurate manner by the boundary conditions. So, the next problem that we are going to deal with would simply state a situation and the statement of the situation would be such that it would, should give you some idea of what exactly is happening inside the system and what you need additionally that is not specified explicitly in the problem.

So, the problem that we are going to deal with is a plate which is moving a thrust, thrust bearing a lubricated plate where the bottom plate is bottom plate is moving with some velocity to the right and you have a lubricant which is present, which is present on top of it, but there exists a stopper over here. So, this is the stopper. The stopper does not allow any liquid to move past that point. So, since that bottom plate is moving. So, it is going to drag, it will try to drag the liquid along with it.



To stop that flow this is what is provided in the problem, that obstruction does not allow any flow past that point and then there is a plate of weight W which is just kept on top of the liquid. Now, the plate will be will sustain the plate will not sink if it is if for some for some reason which we are trying which we will figure out. So, the plate is extremely wide. So, we do not have to worry about the end effects and we also are going to assume that even if two different pressures are going to act at two ends of the plate which is not connected with the stop it is not going to topple. So, if this is a plate and one pressure is at this point the other different pressure is going to act at this point then the tendency natural tendency of any object is going to topple based on which side is at higher, but the assumption of the problem is that even if two different pressures are acting at two ends of the of the plate it can safely be said that an the entire situation can be expressed as if an average pressure of these two is acting at the midpoint which will keep the plate floating.



So, this is a statement of a problem. Now, when we think about it is a simple case of Couette flow nothing else is mentioned except a statement is made that a stopper does not allow any flow to take place. So, how do we incorporate that statement in our governing equation there in comes the understanding part. Now, as you can see in the adjoining figure the entire system can be expressed as if it is a flow between two parallel plates and not only it is a flow between two parallel plates it is there must be an adverse pressure gradient which is present in this situation. And what does this adverse pressure gradient do? It is going to turn some of the fluid turn the fluid around.

So, whatever was coming with the plate is going to go back and it can only go back if the pressure at this point is going to be more than the pressure at this point. So, the problem of having 0 net flow that means, stopper allowing no flow can only be accomplished if we superimpose a Couette flow against an adverse pressure gradient. So, this adverse pressure gradient, the presence of this adverse pressure gradient must be incorporated in the Navier Stokes equation. So, as I mentioned to you sometimes the statement of the problem would be such which you which would let you think that something additional is to be incorporated, is to be considered while writing my governing equation. So, this situation can only be modelled if we correctly understand that there is an adverse pressure gradient in this situation.

And the value of this adverse pressure gradient should be calculated and once we calculate the value of the adverse pressure gradient that would somehow relate to the weight of the plate that is, that is going to be sustained under such situation. So, let us start with our treatment and as you can see I am going to write I have to write the choose the x component of Navier Stokes equation. So, this is an x component of Navier Stokes equation and the positive pressure gradient is nothing, but adverse pressure gradient. As the flow progresses in a certain direction it encounters increasing pressure. So, that is the adverse pressure gradient.

It will still go because there may be something else which is driving it forward. So, in this case the forward movement of the fluid is going to be sustained by the motion of the bottom plate. The equal and opposite backward movement for the no flow situation is created by the creation of an adverse pressure gradient. So, that is all there in this problem. So, when you write the Navier Stokes equation cancel the terms which are not going to do once again since you by now you must be masters of using Navier Stokes equation and cancelling terms.

Suffice to say that there is going to be no v y, no v z, there is going to be v x, v x is not going to be a function of x and v x is going to be a function only of y. So, v x is essentially u in this problem and the gravity is not present, the body force is not present since it is a horizontal system and what we have is only the adverse pressure gradient present in this system. So, this is what the adverse pressure gradient is and once you identify that then you can figure out that at y equals 0 that means, at the top plate at this plate the velocity is no slip condition 0 and at y equals to b that means, on the bottom one the velocity is going to be equal to v 0. So, this is a simple, the entire statement of the problem can be now brought to an equation which is a combination of Couette flow with adverse pressure gradient. We have solved this type of problems before.

$$0 = \mu\frac{d^2u}{dy^2} - \frac{dp}{dx}$$

Adverse pressure gradient

Boundary conditions:

At, $y = 0,\ u = 0$

At, $y = 2b,\ u = V_0$

So, once you do that then this is the expression for velocity that you are going to get where this part is the Couette flow and this part is the pressure driven pressure gradient driven flow. So, and a combination of that is going to give you the net flow rate which it is mentioned that the flow. So, this is the average velocity and once you plug in the expression for u from the from the previous slide then you would be able to obtain what is the, what is going to be the expression for the average value of average expression for the average velocity. Now, it is mentioned that since it stops the flow completely that means, there is no net flow in the system. If there is no net flow in the system then the then the velocity is going to be equal to 0 the flow rate is going to be 0 flow rate to be 0 for that means, the velocity is going to be 0.

$$u = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)[y^2 - 2by] + \frac{V_0}{2b}y$$

So, I equate this, the average velocity to 0 even though I understand that the point velocity can be positive or negative depending on its proximity to the moving belt or the static top plate. So, with that the expression, the expression when you set this expression equal to 0 the value of the pressure gradient would be 3 mu u or 3 mu v naught by twice B square. And you can once again see that the pressure gradient is positive that means, it is the it is an adverse pressure gradient, it obstructs the flow and it and it negates the effect of the Couette flow near the bottle which drives the fluid towards the right whereas, the pressure gradient driven flow moves the fluid to the left and the combination of these two for a specific geometry and specific velocity would be, this is the required pressure gradient to be generated in order to have 0 flow which of course, depends on the gap between the two the velocity of the top plate and the property of the liquid viscosity. Now, the pressure is atmospheric at the other end. So, if you think if we if I go back to this figure once again the pressure over here is atmospheric that means, the gauge pressure at this point is going to be 0.

$$\langle u \rangle = \frac{1}{2b}\int_0^{2b} u \, dy$$

$$\langle u \rangle = \frac{1}{2b}\left[\frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left(\frac{8}{3}b^3 - 4b^3\right) + V_0 b\right]$$

$$\text{No flow: } \langle u \rangle = 0\text{: } \frac{dp}{dx} = \frac{3\mu V_0}{2b^2}$$

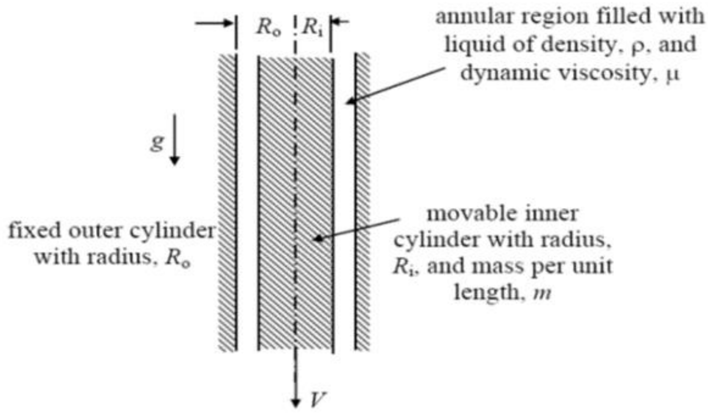
So, gauge pressure once again is nothing, but at most the pressure at any point minus atmospheric pressure. So, the gauge pressure at this point is going to be equal to 0. So, the pressure at this point is equal to 0 and once we do that and once we once we find out what is the average velocity, average pressure in between them then the average pressure is simply going to be 3 by 4 this one which comes directly from this. So, this is the average pressure and you can simply integrate this equation once and put the condition and then you get the average pressure and what is the weight of the plate that can be supported by this pressure it simply averages pressure multiplied by the area which would give you the load per unit width. So, this is the length of the plate this is per unit width.

$$p_{av} = \frac{3\mu V_0}{4b^2} L$$

$$\text{Load/width} = p_{av} \times L \times 1$$

So, this is area multiplied by the average pressure which is acting upwards on the plate would give you the load that this specific system can sustain. So, this is an example as I mentioned is that when through your understanding you incorporate terms which are not explicitly mentioned in the problem. So, understanding reading and understanding the statement problem statement is extremely important. I would urge you to do that before you start writing the governing writing the Navier Stokes equation and simply start cancelling terms. So, that would also be important. So, far we have discussed about boundary conditions before I move into the next problem.

So, far we have thought about boundary conditions which are no slip at the liquid solid interface and no shear at the liquid vapor interface. There can be conditions additional conditions like in the previous problem where the flow rate is going to be equal to 0. In such case the velocity, the average velocity is set equal to 0, but there can be additional conditions which you can figure out by once again by reading the statement carefully. So, the next problem deals with one such situation in which apart from the no slip some other condition is to be incorporated, is to be used, is to be utilized to obtain the velocity distribution or some such thing. So, in this problem statement we have a cylinder which is falling through another cylinder.



So, this is an inner cylinder which is moving down an inner cylinder which is moving down with certain velocity and this is the outer cylinder which does not move. So, it is as if this pane as this pane as the cylinder is falling straight into another cylinder and the space in between the two cylinders are filled with a liquid. So, it is one moving into the other and the assumption is that the cylinder is falling. So, big assumption is that it is falling in perfectly straight and while falling it is not going to tilt in any way such that the gap and it is falling through the centre line. So, the gap in between the moving or the falling cylinder and the fixed or the outer cylinder remains unchanged a big assumption.

But let us say for the sort the solution of this problem we would we would assume that this is the situation. No toppling, no sideways movement of the of the falling cylinder, gap remaining the same. So, the cylinder which is falling has a mass per unit length m. So, the total mass of the cylinder would simply be equals to m times L where L is the length of the, where L is the length of the of the of the cylinder. Now, there is no pressure gradient, no pressure gradient within this system and where and there is no swirl velocity component.

That means, there is no movement in this direction, the motion of the film is always directed down. So, it is going to initiate a downward velocity, but no velocity in the theta direction or in the r direction. It is only velocity is in the z direction along with the freely falling cylinder. So, with this we need to figure out what is the vertical speed of the inner cylinder as a function of the parameters, the gravitational constant the two radii inner and the outer the mass per unit length of the cylinder the density and the viscosity. And it is mentioned here, specifically mentioned here that the space between the two cylinders is not too small.

It is not too small compared to the radii of the cylinder. That means, I must use cylindrical, I must use cylindrical coordinate system and I cannot use the assumption that the gap is so small. So, a Cartesian coordinate system can be used. So, a problem of one cylinder falling freely into another through another cylinder with a liquid in between with no sideways bend the gap between the two cylinders remaining the same and we have to figure out what is going to be the vertical speed, the final vertical speed v of the inner cylinder. Now, you have you probably know I can give you an example which with which you must be familiar with that when you let a ball fall into a liquid where the density of the ball is higher than the density of the liquid then it starts its downward motion.

As it starts its downward motion there will be forces opposing forces acting on it. So, the gravity downward buoyancy upward and there is going to be a drag force because of the relative velocity between the ball and the liquid that is also returning the motion of the downward motion of the ball. So, initially the ball would start accelerating, as it starts accelerating the value of the gravity and the buoyancy force will remain unchanged, but the upward acting drag force will be will keep on increasing because that drag force is a function of velocity. So, once the drag force reaches a certain value in which all three forces are balanced, that means, the gravity, the upward buoyancy and the upward acting drag forces are balanced then from that point onwards the ball will have a constant velocity with which it is going to settle. So, that is known as the terminal velocity of the spherical particle.

So, in here the cylinder is also going to have a constant velocity once that condition is reached. So, we will have to figure out what it is going to be. So, for the inner cylinder moving at a constant velocity the downward force is exactly balanced by the viscous force. So, the viscous force is tau w what is the wall shear stress multiplied by A w at the inner cylinder. So, the upward downward acting gravity is going to be exactly balanced by the viscous shear force.

$$(\tau_w A_w)\big|_{Inner\ Cylinder} = m\,L\,g$$

So, that is the condition, that is an additional condition which one can obtain by understanding the physics of the problem. So, with this let us move on and try to figure out, try to find out how this problem can be solved. Once again it is a z component of equation of motion which is in cylindrical coordinate that we must consider and then we need to cancel the terms that it is at steady state, it is at steady state. So, this part would not be there is no v r and v z are not a function of theta and v z is not a function of z, there is no applied pressure gradient and v z once again not a function of theta z and there is going to be a rho g and there is going to be this term since v z is a function of r. So, those two terms will remain in the governing equation and this governing equation can now be integrated and this is going to give rise to a distribution which with a logarithmic term in there.

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

So, what are the two boundary conditions? The two boundary conditions are velocity is equal to capital V yet to be determined we do not know what this v is going to be, but v z is going to be capital V at the inner cylinder at r equals r i which is falling with some velocity. And the second boundary condition is v z equals 0 due to the static nature of the outer cylinder at r equals r 0. So, both are no slip conditions, one in which case the velocity is equal to the velocity of the downward velocity of the cylinder and the other is the static cylinder the velocity at the liquid solid interface at that point would be equal to 0. So, this these are the two equations which would let you figure out what is going to be the expression for c 1 and c 2. Once you do that then the complete expression for velocity would be something like this, where the integration constant one of the integration constant c 1 would simply be this one.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{\rho g}{\mu}$$

$$v_z = -\frac{\rho g r^2}{4\mu} + C_1 \ln r + C_2$$

Boundary Conditions

$v_z = V$ at $r = R_i$ => $V = -\frac{\rho g}{4\mu} Ri^2 + C_1 \ln R_i + C_2$

$v_z = 0$ at $r = R_o$ => $0 = -\frac{\rho g}{4\mu} Ro^2 + C_1 \ln R_o + C_2$

So, it is a cumbersome it is a big relation and this also underscores that if we could if we could have used the Cartesian coordinate system the expression would be more compact it would not contain any logarithmic term and it would be easy to work with. But it has been specifically mentioned that the gap is not small with respect to in comparison to the radii of the cylinder. Therefore, we do not have the liberty or the luxury of assuming that it is a Cartesian coordinate system. So, with this then you in next step is to figure out what is going to be the shear stress. So, in order to obtain the shear stress because the shear stress when multiplied by the area would give us the opposing force against gravity with which the liquid with which the cylinder is falling.

$$V = \frac{\rho g}{4\mu}\left(Ro^2 - Ri^2\right) + C_1 \ln \frac{R_i}{R_o}$$

$$C_1 = \frac{1}{\ln \frac{R_i}{R_o}} \left[V - \frac{\rho g}{4\mu}\left(Ro^2 - Ri^2\right)\right]$$

$$\frac{dv_z}{dr} = -\frac{\rho g r}{2\mu} + \frac{C_1}{r}$$

$$\tau_{rz} = \mu \frac{dv_z}{dr} = -\frac{\rho g r}{2} + \frac{C_1 \mu}{r}$$

So, we figure out what is the velocity dv z dr and we also underscore that v z is a function only of r. So, there is no need to think about additional terms in the expression for tau. So, dv z dr is simply going to be this the combination of these two terms and tau would be mu times dv z dr plus mu since this is the force acting on the on the cylinder. And so, you would be able to write then convert the equation the idea that force on the cylinder must be equal to force due to gravity. So, the viscous drag force on the cylinder must be equal to the force due to gravity.

$$\tau\big|_{r=Ri}\, 2\pi R_i L = m\, L\, g$$

So, if you do that then what you would see is that you need to figure out what is tau at r equals r i that is at the inner cylinder and so, multiplied by the area of the inner cylinder which is twice pi r i times L must be equal to m times L because m is mass per unit length. So, that is why I multiply it with L, the length of the length of the of the cylinder and g. So, the shear stress multiplied by the area must be equal to the mass per unit length times length times g. So, this is the additional condition that we need to incorporate to obtain my final expression. So, upon substitution of the expression for the shear stress we would get this complex relation for the unknown velocity, unknown constant velocity with which the with which the inner cylinder is falling in the liquid.

$$V = R_i \ln \frac{R_i}{R_o} \left(\frac{\rho g R_i}{2\mu} - \frac{mg}{2\pi R_i \mu}\right) - \frac{\rho g}{4\mu}\left(R_i^2 - R_0^2\right)$$

So, what did we learn in through the from this problem is that sometimes one has to think of additional relations which would be necessary to solve a specific problem. And when to use the Cartesian coordinate approximation and when you cannot use that as it has been mentioned that the gap is not too small you cannot use the Cartesian coordinate transformation from a cylindrical coordinate system. And secondly, the no slip conditions are going to be valid on the moving cylinder and on the stationary cylinder. Additional consideration, additional constraint would come in the form of balancing of gravitational force with that of the oppositely acting viscous force. How do I calculate viscous force? The first step in calculating the viscous force is to know the velocity distribution.

The velocity distribution at any given location multiplied by mu, the velocity, it is from the velocity distribution you can find out what is the velocity gradient. Velocity gradient multiplied by mu the viscosity at that specific location would give you the value of the shear stress. So, shear stress multiplied by the area would give you an additional idea of the force present in the system. So, with this additional condition we have obtained this expression for the velocity for a moving fluid in such situations. So, this is a very good example of introducing physical insight into the system for final solution.

So, both the problems which we have solved in this class require additional thinking beyond just simply cancelling the terms. In the first case you had to think of an adverse pressure gradient which would cause the some of the fluid to move in the other direction. And in the second problem an additional force balance is to be incorporated to make them to make the situation moving at steady state with no variation of velocity with time, but both required approximations. In the first case the plate must be kept horizontal even though dissimilar forces are acting at two ends which is which is difficult to obtain. And in the second case we are assuming that the cylinder is falling through the central line at a constant velocity without any sideways movement.

So, those are approximations to the real situation, but it would give us very good insights into the physics of the process. So, that is all for this class we will meet again with some more examples of Navier Stokes equation. Thank you.