**Momentum Transfer in Fluids**
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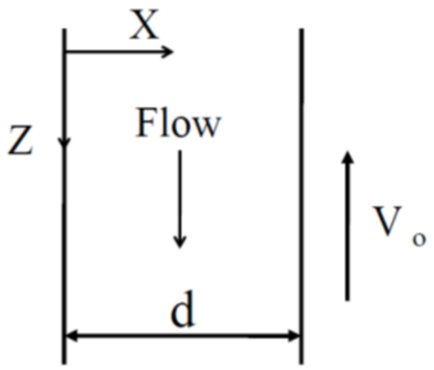
**Week-04**
**Lecture-19**

We will continue our studies on the use of Navier-Stokes equation for problems which can easily be solved by choosing the right component of the Navier-Stokes equation, cancelling the terms that are not relevant and thereby obtain the governing equation. And we already are aware of what could possibly be the boundary conditions, the common boundary conditions in problems involving fluid mechanics momentum transfer. So, in this class or in subsequent classes also I will try to solve different types of problem which would give you some idea, something new about the solution methodology and not only solution methodology, they would let you understand how to express the physics of a statement in terms of an equation. So, our first problem today is about flow between two parallel plates. So, if you look at the figure there are two plates very long plates where you have a liquid in between them and the liquid has certain density and viscosity. Now, one of the plates, this plate, is kept stationary while the plate at x equals d it start, it is moving up with some velocity v naught.




Now, you understand that if I did not have any body force acting in this case, it would simply be a problem of Couette flow. But over here I have the gravity which is acting downwards and the upward motion of the plate drags the liquid along with it. So, what is going to be the draining rate that we would like to figure out? So, the first step is therefore, to obtain the velocity profile and by the careful combination of the gravity force I mean obviously, the gravity force in this case is a constant. So, if you keep on increasing the velocity of the right-hand side plate then the liquid will start the flow of the leak upward flow of the liquid will be enhanced.

So, a point may reach at which the downward flow of the liquid due to gravity would be exactly balanced by the upward flow initiated by the motion of the top motion of the right-side plate. So, second problem is what is the value of v naught going to be that would ensure that you have a 0-flow rate. And the third one is identifying the location of the 0-shear plane. Now, this is probably the most important part of it, because right now I have the liquid and the solid and here the liquid and the solid interface. So, at the liquid solid interface I know that my boundary condition would be that the relative velocity would be 0, the no slip condition.

Since we I do not have any liquid air interface there is no question of having a 0 shear plane in this, but we would also have to think that the some flow is moving up some flow is coming down. So, there may exist a 0 shear plane in between the 2 plates and our job is to find out what is the location of this 0 shear plane. So, we will start first with the Navier Stokes equation and by now you can clearly figure out that this is a Cartesian coordinate system. And since the flow is in the z direction so, it is the z component of Navier Stokes equation which we must write and then we have to cancel the terms which are not going to be relevant in this specific situation. So, I have already deleted or rather I have already crossed out the terms which are not going to be important in this case.

For example, the first term that is del vz del t, it is a steady state process, steady state situation. So, there is no variation of velocity with respect to t. I also do not have; it is a one-dimensional flow. So, there is no flow in the x direction or in the y direction. So, therefore, the second and the third term would be 0 because v y and v x would be 0.

And finally, for the last term on the on the left-hand side, the velocity is fully developed and we do not have any variation of v z with z. So, therefore, this term also going to be 0. It is a case so; the entire left-hand side of the Navier Stokes equation is 0 which essentially tells us that the convective contribution to flow there is no convective momentum transport in this specific case. When you move to the right-hand side, I have the pressure gradient, the imposed pressure gradient in the system. It is a freely falling liquid with one plate stationary the other plate moving up.

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

So, since there is no applied pressure gradient that term would also be equal to 0. When we come to the next and the most important term over here v z if you look at the figure once again v z is going to be a function of x. The value of the v x is going to be 0 at x equals 0 that means, because of no slip condition over here and the velocity is going to be that of the moving plate which is equal to v naught. So, the in between these two velocities the velocity v z varies with x. So, I cannot neglect that term whereas, v z is not a function of y, v z is not a function of z and therefore, we can discard these terms from the governing equation.

$$\rightarrow \mu\frac{d^2 v_z}{dx^2}+\rho g=0$$

And of course, I have a nonzero, the gravity is going to be there and as far as per the coordinate system that we have drawn here g suffix z is simply going to be equal to g. So, with this the and so, the explanation that I have given are provided over here and therefore, the governing equation becomes this term and these two terms would be the governing equation. Now, this equation the boundary conditions as I have already mentioned are no slip at these two points only thing that I would like to draw your attention to is that v z as per the coordinate system that we have drawn over here v z is equal to minus v 0 at x equals d. Since, the velocity since the fluid the plate is moving up with the, with this one would be able to obtain what is the final form of the velocity. I am not doing step by step analysis from that equation which you should do.

Boundary conditions: $v_z = 0 \; at \; x = 0$; $v_z = -V_0 \; at \; x =$d

$$v_z = \frac{\rho g d^2}{2\mu}\left[\frac{x}{d} - \frac{x^2}{d^2}\right] - \frac{V_0 x}{d}$$

So, once I have the velocity then comes the second part of the problem. The second part of the problem tells us that what is going to be no flow condition what value of v naught, v naught must be in order to create a situation in which there is no flow, no net flow. So, whenever we talk about net flow, we first must figure out what is the average velocity and then set the average velocity to 0 which would give us the unknown v naught necessary to have a flow, 0 net flow. So, the average velocity it is averaged over the flow area and the flow area is v z d x where x varies from 0 to d. There is also going to be because this is this is per unit width.

For no flow, $\langle v_z \rangle = \frac{1}{d}\int_0^d v_z dx = 0$

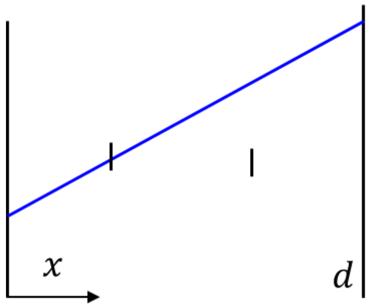
So, I can have another integration sign where the area is simply going to be v z d x d y. So, it should be v z d x d y and double integration, the x varies from 0 to d and the y varies from 0 to some width of some width of the plate and over here I have 1 by d times w. So, this is the definition of the average velocity, but we understand that v z is not a function of y and if the plates are wide. So, even if I drop that that part it will not make any difference. So, this is the expression for the average velocity once you plug in the expression of v z that we have obtained in the previous slide what we have obtained in the previous slide is this is the expression for the velocity v z in terms of x.

$$\rightarrow v_0 = \frac{\rho g d^2}{6\mu}$$

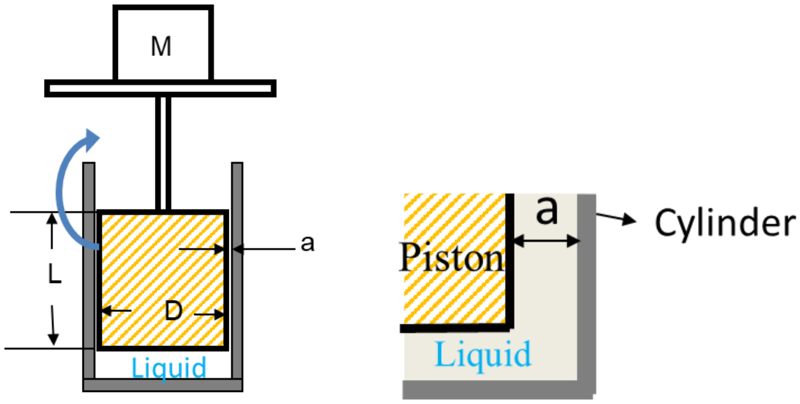
So, if you plug that in over here what you get is v naught the upward moving velocity will simply be rho g d square by 6 mu. So, that is the velocity the plate should have to obtain a 0 net flow and when the boundary condition we have already taken v z is equal to minus v 0. So, it is it is clear now that we have to have an upward velocity with a magnitude equals to rho g d square by 6 mu. Coming back to the next part last part of the problem we need to figure out what is going to be the shear stress distribution in the small space in between the two plates and the location of the 0-shear plane. So, this is my starting point where I have the velocity expression that we have already derived from the velocity, from the velocity and the we can we can also be would be able to obtain what is going to be the expression for tau.

$$\tau_{xz} = -\mu \frac{d^2 v_z}{dx^2} \quad \rightarrow \tau_{xz} = \rho g x - \frac{\rho g d}{2} + \frac{\rho g d}{6}$$

For $\tau_{xz} = 0, \; x = \frac{d}{3}$



So, this is the expression for tau and when you set this tau x z to be equal to 0, you would see that the that the shear stress is going to be 0 at this location and your solution would give you the value of x to be equal to d by 3. So, for a distance, at one third distance from the static plate, you would get a 0 in the shear stress. So, this is this problem again deals with a situation where you choose the Navier Stokes equation use the boundary condition, get the velocity distribution, get the average velocity set the average velocity to 0 in order to figure out what is the upward moving velocity upward moving velocity of the plate has to be in order to create a condition of no flow and then you figure out what is tau x z and from the expression of tau x z you can see that there exists a 0 shear plane some point in between and the location is x equals d by 3. So, that is this problem. We are going to talk; we are going to have a different type of problem in this one.




So, this problem it is a piston cylinder apparatus which is quite common. So, in practice so, you have a piston that goes into a cylinder and since they you need to have as small friction as possible to reduce wear and tear of the piston or the cylinder you use a lubricant. So, the lubricant creates a very thin film between the snugly fit the piston and the cylinder assembly. So, what is shown here by this this would by the by this a that is the small gap in between a piston which is cylindrical in shape and the cylinder. So, there is a liquid, a lubricant which is kept in between them and we will try to figure out what is going to be the leakage rate of this lubricant when the piston comes down.

Now, we are when the piston comes down, the purpose of the piston coming down is to create high pressure in the small space in the in the in the space between the cylinder between the bottom part of the cylinder and the piston. So, this you can you can generate high pressure by this piston cylinder assembly, but the engineering application of fluid mechanics in this is you need to have a lubricant and as an engineer you would like to find out what is going to be the leakage rate of the lubricant. Because you would like this portion, the interacting portion to be to be to be filled up with a thin film of liquid all the time. So, it is a cylindrical coordinate system, but I have mentioned before that if the gap in between the two surfaces is very small compared to the radius of the cylinder, compared to the radius, then the cylindrical coordinate can be thought of can be can be simplified to a Cartesian coordinate system. So, think about, this is the piston this is the cylinder they are very close to each other the curvature if you think about the curvature and the gap in between them if it is so, then you can you can simplify this problem as if they are two parallel plates situated by a small distance apart with a liquid with the lubricant in between.

So, a cylindrical system can be simplified to a Cartesian coordinate system if the gap in between them is extremely small which is satisfied in a piston cylinder apparatus, piston cylinder assembly. So, we so, this is the problem therefore, is there are two parts of the problem that we need to find out what is this value of mass m that can create a pressure of 1.5 mega Pascal and this is the gauge pressure. So, the absolute pressure is gauge pressure plus atmospheric pressure which is generally of the order of 1 into 10 to the power or 1 into 10 to the power 5 Newton per meter square, but the pressure over here that needs to be generated is 1.5 mega Pascal, one order of magnitude higher than the atmospheric pressure.

So, this is about 15 roughly very roughly 1.5 into 10 to the power 6 so, 15 atmospheric pressures. And secondly, we need to figure out what is the leakage flow rate as a function of radial clearance. So, we need to find out that for this radial clearance how much of lubricant we are going to lose when the piston starts coming down. And the oil which is look which is situated in between the viscosity and the density of the oil is provided and it we also must figure out the maximum allowable radial clearance.

So, that the vertical movement of the piston is going to be less than 1 millimetre per minute. So, for that pressure generated by placing the mass on top of the assembly what is going to be the vertical clearance, what is going to be the clearance which is A that would allow a movement only of 1 millimetre per minute. So, that value of A, the specific value of A must be had to be evaluated. So, this problem is a fine example of use of Navier Stokes equation in a problem of industrial importance. So, we will start from the basics.

The first point is that obviously, the gap is very small in comparison to the radius or the diameter D of the cylinder. If that is so, the radial coordinate system can be converted to a Cartesian coordinate system which is easier to handle in comparison to that of that of the other system either radial or spherical system. So, this is the figure that I would like you to take a note of and then also you see the location of the coordinate system where the origin of the coordinate system is. Now, first find out the pressure. So, we know that the pressure is simply going to be equals or rather the force due to pressure is going to be equal to m times g and this p the pressure that is to be generated is given as 1.5 into 10 to the power 6 plug in the value and what you would get is m equals 4.32 kg. So, by placing a mass of 4.32 kg at the top you are generating a pressure of 1.5 into 10 to the power 6 and Newton per meters in Pascal in the space below the cylinder.

$$\frac{\pi D^2}{4}(p - p_{atm}) = Mg$$

$$M = 4.32 \text{ kg}$$

So, the next comes something which I have described that it is essentially then a flow between parallel plates separated by distance a and the width of the plate in this case since it is a cylinder 1 and you have opened it up. So, the width of the plate is simply going to be equals pi d and the length as specified in the figure, the length is going to be equal to L. So, with this now we start solving this using Navier Stokes equation. So, there is an understanding part involved as well as there is going to be now going to be the application of Navier Stokes. Now, once again the velocity is the motion is in the y direction.

So, there we need to write the y component of the Navier Stokes equation which is which you can see over here. So, the first thing is it is a steady state problem. So, there is not going to be any transient one there is no v x the v x there is no velocity in the x direction. So, the second term disappears the velocity v y is not a function of y. So, therefore, the third term will also be cancelled and the fourth term the v y is the v y is not a function of z or the v z is equal to 0.

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + \cancel{v_x \frac{\partial v_y}{\partial x}} + \cancel{v_y \frac{\partial v_y}{\partial y}} + \cancel{v_z \frac{\partial v_y}{\partial z}}\right)$$
$$= -\frac{\partial P}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \rho\, g_y$$

So, this is going to give you the entire left-hand side equal to 0 no convective transport of momentum. And in this case, I have a pressure gradient as a massive pressure gradient which is present in the y direction. So, it is 1.5 into 10 to the power 6 on one side and 1 into 10 to the power 5 on the other side. So, therefore, we cannot neglect the first term, this term will be will remain in my governing equation and at the same time v y is a v y is a function of x.

So, this term will also have to be retained v y is not a function of y. So, this term will be cancelled and similarly v y is not a function of z. So, that can also be also be cancelled, but I must have this g y, the body force the gravitational body force which is present in this system. So, the boundary conditions are at x equals 0 and at y equals at x equals a is going to be the no slip conditions at 0 and at v. So, this one plate is stationary one plate is stationary that is at x equals 0 and the other plate which is moving.

Boundary conditions:

$$\text{At, } x = 0,\ v_y = 0$$

$$\text{At, } x = a,\ v_y = V$$

So, that is has a constant velocity equal to capital V. So, with this you should be able to solve this is your governing equation and once you look at the governing equation many a times in engineering calculations you get an equation, you do not start solving it immediately because there can be substantial simplification can be obtained by evaluating the order of magnitude of each term. So, what we need to figure out here is what is causing the flow, the pressure gradient, the gravity, what is going to be the relative effect of each one of them. Is it a situation in which the pressure gradient driven flow is so large that it would completely overshadow the effect of gravity. So, this kind of considerations would let you simplify the Navier Stokes equation or your governing equation further.

Now, if you think if you think the pressure gradient is going to be very large and the gravity effect may not be that large. So, let us do a quick order of magnitude analysis not exact order of magnitude analysis and see if I can say if we can say something about the second and the third term or the relative importance of the second and the third term. So, that is that is what we have done over here. Now, rho and g rho times g rho are of the order of 10 to the power 3 right.

Quite small compared to $\frac{dp}{dx}$

$\rho \rightarrow o(10^3)$ , $g \rightarrow o(10)$ , $\rho g \rightarrow o(10^4)$

$\frac{dp}{dx} = \frac{1.5\times10^6}{25\times10^{-3}} \approx o(10^8)$

So, rho of water is 1000 kg per meter cube. So, it is order is going to be of the order of 10 to the power 3 g has order of equal to 10. So, rho times g will be of the order of 10 to the power 4. Now, what is dp dx in this specific case? The pressure difference is 1.5 into 10 to the power 6 whereas, dx the distance is 25 millimetre I mean the length of the gap in between the length. So, it is dp dx is going to be of the order of 10 to the power 8.

So, 1 is of the order of 10 to the power 4, the other is of the order of 10 to the power 8 which clearly tells me that this is a situation which is principally governed by pressure gradient and not by gravity. So, I can safely drop the second term from my governing equation. So, this type of simplification many a times would allow to allow us to deconstruct a difficult equation into something more manageable. We will have more examples later. So, this is my governing equation then and this equation when it is integrated it would give the this form and the boundary conditions are at x equals 0, v y equals 0 and at x equal this dp dx essentially this is this is dy, this should be dy because the y the length of the if you go back to the if you go back to the statement of the problem the length of the piston I think is given as 25 millimetre.

$$\mu\frac{d^2v_y}{dx^2} + \cancelto{0}{\rho g} - \frac{dp}{dx} = 0$$

$$\mu\frac{d^2v_y}{dx^2} = \frac{dp}{dx} = \frac{\Delta p}{L}$$

$$v_y = \frac{1}{2\mu}\frac{\Delta p}{L}x^2 + C_1x + C_2$$

Boundary conditions:

$$\text{At, } x = 0,\ v_y = 0 \ \rightarrow C_2 = 0$$

$$\text{At, } x = a,\ v_y = V \ \rightarrow C_1 = \frac{1}{a}\left[V - \frac{1}{2\mu}\frac{\Delta p}{L}a^2\right]$$

The piston length is 25 millimetre and of course, the pressure gradient is acting in the in the y direction. So, in here the pressure gradient is acting in the y direction. So, pressure is more over here 1.5 into 10 to the power 6 and at the top the pressure is going to be equal to atmospheric.

So, the dp dx essentially is dp dy. So, this is dp dy there is a typo over here this x should be equal to y. So, with this now I would be able to obtain what is the functional what is the functional form of v y which can then be you can find out figure out easily what is the velocity in this case and once you have the expression for velocity from the governing equation and using the boundary conditions you should be able to obtain what is the average velocity because ultimately we are going to figure out what is the what is the leakage flow rate. Now, to obtain the leakage flow rate, the average velocity for average velocity of the liquid in the small gap in between the cylinder and piston will have to be multiplied by the area available for flow. It is the average velocity times area is essentially the flow rate which in this case is the leakage flow rate of the of the lubricant.

$$v_y = \frac{1}{2\mu}\frac{\Delta p}{L}\left[x^2 - \frac{x}{a}a^2\right] + V$$

$$\langle v_y \rangle = \frac{1}{a}\int_0^a v_y dx = -\frac{1}{12\mu}\frac{\Delta p}{L}a^2 + \frac{Va}{2}$$

$$\langle v_y \rangle = -\frac{1}{12\mu}\frac{\Delta p}{L}a^2 + \frac{Va}{2}$$

So, this v y this is the expression for v y. Now, once again one must take in consider which term is going to predominate is it going to be the first term or the second term. One contains A A the other contains A square. This v is very small, few millimetres 1 millimetre per minute, but this delta p by L it is extremely large as we have seen in the previous slide. So, we can safely say that v y is can be is not is approximately equal to minus approximately equal to the contribution to the first term. So, the upward movement upward movement of the or rather the movement of the liquid because of Couette type of flow is going to be significantly smaller than the flow due to the pressure gradient.

$$\langle v_y \rangle = -\frac{1}{12\mu}\frac{\Delta p}{L}a^2$$

Again, $V = 1\ \mathrm{mm/min}$, 2nd term on RHS is quite small

This approximation lets us solve the problem rather easily if it is not if I had to retain the second term then it would lead to a quadratic equation and it the calculation becomes more slightly more complicated. So, using your concept of which flow is going to be important you can safely see that the second term will have minimal role in the calculation of the average velocity or the average flow rate. So, with this the flow rate is simply going to be the average velocity multiplied by the area which is which is available. So, the area available for flow is simply going to be pi times D times A where A is the small gap in between the two. So, that is the final expression for the average flow rate and the flow is in the negative y direction as it should be because of the high pressure at this point the flow essentially is going to be in the upward direction and upward direction as per the coordinate system chosen by us is going to be negative.

$$Q = \langle v_y \rangle\, a\, \pi\, D = -\frac{1}{12\mu}\frac{\Delta p}{L}a^3\pi D$$

The flow is in the negative y direction as it should be

For downward movement (V m/s) the volume displaced is,

$$Q = \frac{\pi D^2}{4}V = \frac{\pi}{4}(0.006)^2 \times \frac{0.001}{60}\frac{m^3}{s} = 4.71 \times 10^{-10}\,\frac{m^3}{s}$$

$$Q = -\frac{1}{12\mu}\frac{\Delta p}{L}a^3\pi D \text{ , Thus, } a = \left[\frac{12\mu Q L}{\pi D \Delta p}\right]^{1/3}$$

With $\mu = 0.42\ \frac{\mathrm{Ns}}{\mathrm{m}^2}$

$$a = \left[\frac{12}{\pi} \times \frac{0.42 \times 4.71 \times 10^{-10} \times 0.025}{0.006 \times 1.5 \times 10^{6}}\right]^{1/3} = 1.28 \times 10^{-5}\mathrm{m}$$

And that is why the flow rate is going to be negative for the coordinate system that we have used in this. So, then the next part is simple because in this I need to figure out what is going to be the velocity and for that velocity is essentially downward movement of 1 millimetre per second. So, we can calculate what is the flow rate and this flow rate can then be equated to the flow rate that we have calculated that we have expressed in terms of Navier Stokes equation. So, my q is simply this part and then this would once you once you do this plug in the values and you can figure out the exact value of A to be equals to 1.28 into 10 to the power minus 5 meters. This once again underscores the fact that this gap is so thin that our initial assumption of changing the cylindrical coordinate system to a Cartesian coordinate system is well justified. So, this example then talks about converting a cylindrical coordinate to a Cartesian coordinate system looking at each term of the Navier Stokes equation separately to see whether one predominates over the other, which would allow us to simplify the Navier Stokes equation the governing equation even further for solution. So, that is the main message of the last problem that we have discussed. We will continue doing looking at problems of different type in our next classes. Thank you.