Momentum Transfer in Fluids
Prof. Sunando DasGupta
Department of Chemical Engineering
IIT Kharagpur
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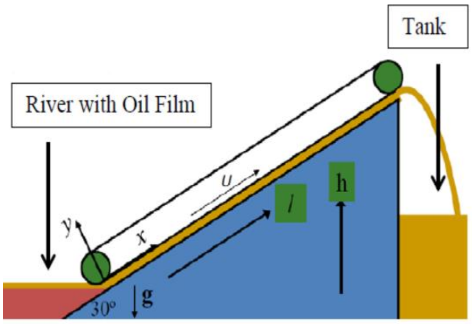
Lecture-18

We are going to continue once again with the application of Navier Stokes equation. But this time we will choose problems which I would say there are there is some there are some practical applications to it. So, we would, the first problem that we are going to show, going to start with is the problem of a unique problem in which there is some oil spill in a river. And the oil spill is going to be tackled by some novel innovative idea what is what is being done is that the there is a boat with an inclined plane as you see over here. This is the inclined plane of the boat this is what the inclined plane is, as the boat moves in the river what you see as the yellow film that is the oil the red one is the water. There is a belt which moves in the upward direction.



So, it will drag the oil film along with it across this and then it is going to get deposited on, in a tank on the boat. So, you have a stationary boat surface and the belt is moving upwards. It is dragging the oil and then it is going to get it into a tank on the boat. So, the it is almost like a Couette flow in which one of the surfaces is moving one of the solid surfaces is moving the other is almost stationary. However, there is no applied pressure gradient, apart from that there is a presence of a presence of a body force in the reverse direction.

So, gravity tries to pull the film back to the river whereas, the upward motion of the belt forces the film to go over the top into the tank. What we need to figure out is what is the force the discharge of oil that means, how much what is the flow rate at which the oil is going to get deposited into that in the tank. What is the force acting on the belt, how much of what is what kind of a force is needed to move the belt and the power required to move the belt. One point of caution here is that there would be additional losses which the machine that drives the belt has to overcome which are frictional losses be in the moving machinery. We are not getting into that part we are only trying to calculate what kind of power is needed to overcome the fluid friction to make the viscous fluid travel up against gravity into the tank.

So, we are going to start once again with Navier Stokes equation. The point is which Navier Stokes, which component of Navier Stokes equation that we need to use and what are the what are the factors that will make me will allow me to simplify the Navier Stokes equation into the governing equation. The origin is on the boat surface. So, this is where the origin is and we are will it will give us the appropriate boundary conditions. But let us first start with the proper component of the Navier Stokes equation.

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]+\rho g_x$$

Here in you can see that the motion is only in the x direction, there is no flow in the y or in the z direction and it is a Cartesian coordinate system. Therefore, we are going to write the x component of Navier Stokes equation in Cartesian coordinate system and start from there. So, this is the x component of the Navier Stokes equation. Let us see what kind of simplifications we can incorporate based on the understanding of the flow physics for this specific problem. The first one that it is a steady state.

So, this term would disappear. There u the x component velocity is not a function of x, it is a function of y, but not a function of x. So, that term would disappear. There is no component y component of velocity. So, v is going to be equal to 0, no z component of velocity.

So, w is going to be 0. It is a situation in which we do not have any imposed pressure gradient in the system. So, dp dx is going to be 0. The velocity x component of velocity is not a function of x, it is a function of y only. So, this term I cannot neglect and would remain in the Navier Stokes equation.

$$\frac{d^2u}{dy^2}=\frac{\rho g}{\mu}\sin\theta$$

u is not a function of z. So, this term would disappear as well and there is going to be somebody force which will try to drag the fluid back into the river, the oil back into the river. So, these are the two terms which will constitute the governing equation for this specific situation and therefore, my governing equation for this would simply be the terms which I have circled over there. Now, this now this governing equation would have to be solved with the appropriate boundary conditions. So, let us see what are the appropriate boundary conditions.

BC 1 At $y = 0$ $u = 0$

BC 2 At $y = h, u = U$

At y equals 0 that means, on this plate on the movement on the boat surface the velocity is going to be 0 due to no slip condition. And on at y equals h that means, on the moving belt surface, the velocity of the fluid would simply be equal to the velocity of the belt which has been shown to be equal to capital U. So, the two boundary conditions are at y equals 0 small u, the velocity of the fluid is going to be 0 no slip and at y equals h small u the velocity of the fluid would be equal to capital U the velocity of the belt. So, with this one can proceed integrate it once and to integrate it once to obtain one integration constant and then another integration constant A and B. So, when you apply the boundary conditions you can find that B is going to be 0 and A has this specific expression.

$$\frac{du}{dy}=\frac{\rho g sin\theta}{\mu}y+A \quad and \quad u=\left(\frac{\rho g sin\theta}{\mu}\right)\frac{y^2}{2}+Ay+B$$

Applying the BC s ➔ B = 0 and $A=\left(\frac{\rho g sin\theta}{\mu}\right)\frac{h}{2}+\frac{u}{h}$

So, once you put back these into the governing equation, the equation for U the velocity of the oil in the thin film formed between the boat surface and the belt, the upwardly moving film is going to be this. So, if you recall from our previous analysis this essentially is the contribution of Couette flow. This is due to the motion of the fluid motion of the oil initiated by the upward moving, upward moving belt. Whereas, this part is due to the flow due to the flow initiated by the gravity which is acting in the reverse direction. So, there are two parts to this flow and depending on what is the value of theta, what is the value of mu and the value of u the flow net flow could be in the upward direction.

$$u = -\left(\frac{\rho g sin\theta}{\mu}\right)\left(\frac{hy}{2} - \frac{y^2}{2}\right) + \frac{Uy}{h}$$

So, the now we have to find out the volumetric flow rate that was the first part of the problem. The volumetric flow rate can be obtained once I integrate this velocity over the entire flow area. Now, if you recall from this figure the flow area is simply going to be del d y times d z integration over the limit. So, the volumetric flow rate per unit width so, I am not worried about z anymore. So, it is the per unit width and it simply must since u is not a function of z whatever be the width of the of the of the belt you can simply multiply the final expression with that to obtain what is the total flow rate.

$$Q = \int_0^h u dy = -\int_0^h \left[\left(\frac{\rho g sin\theta}{\mu}\right)\left(\frac{hy}{2} - \frac{y^2}{2}\right) + \frac{Uy}{h}\right] dy = -\frac{\rho g h^3}{12\mu} sin\theta + \frac{Uh}{2}$$

But so, but we are what we are finding here is the flow rate per unit width. So, we will integrate it between 0 to h where h is the separation in between the boat surface and the belt u dy. So, you plug in the expression of u in here integrate this and what you get is this as your volumetric flow rate. Once again, I point out that the second part is due to the Couette flow whereas, the first part is due to the gravity driven flow. The algebraic sum of this would give me the whatever be the total volumetric flow rate for this specific case.

$$Q = -\frac{860\ x\ 9{,}81\ x\ 0.002}{12\ x\ 10^{-2}} \sin 30 + \frac{3\ x\ 0.002}{2} = 0.0027 \frac{m^2}{s}\ (per\ unit\ width)$$

For the width of 5 m, the volumetric flow rate will be equal to 0.0135 m^3/s.

So, with this knowledge now we can proceed to obtain the numerical value of q by putting in all these numbers while understanding that it is going to be per unit width and since it is per unit width that is why the unit is in meter square. So, the volumetric flow rate and for a width of 5 meter which has been specified I simply multiply this term with 5 to obtain this flow rate in meter cube per second. So, it is a straightforward application of Navier Stokes equation resulting in the flow rate. The next is what we must figure out is what is the what is the force on the belt. Now, the force on the belt is due to shear stress.

Evaluate $\tau = \mu\ du/dy$ at the moving belt from the expression for velocity as:

$$\frac{du}{dy} = -\left(\frac{\rho g sin\theta}{\mu}\right)\left(\frac{h}{2} - y\right) + \frac{U}{h}$$

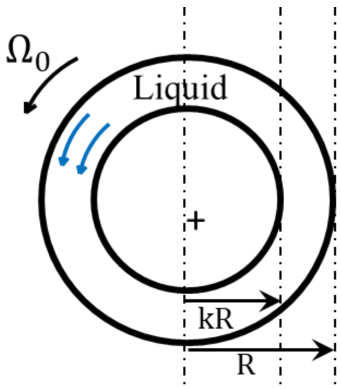
At the moving belt, $\tau = \mu\left(\frac{du}{dy}\right) y = h = \left(\frac{\rho g sin\theta}{2}\right) h + \frac{\mu U}{h}$

It is the shear stress exerted by the fluid by the oil which would try to retard the motion of the belt. So, therefore, tau is simply going to be equals to mu the viscosity times velocity gradient that would give me the shear stress. I already have an expression for du d y. So, I plug that u in here to obtain what is going to be my d u d y and from there I put in the expression of d u d y. I understand that my y in this case is equal to h that is my this the coordinate system is on the plate of the boat and y is the separation between the two.

So, when I am trying to figure out what is the shear stress, I am on the belt I am evaluating the shear stress at y equals h. Once I do that then this becomes the expression for my shear stress. After I have the and we can put the values that are known to us the thickness of the film, the viscosity and the velocity of the belt. So, you can figure out what is the tau at the belt the shear stress at the belt to be equal to 19.21Newton per meter square.

You can you should do it on your own and check that whether you are getting the same value and this would also be a good practice for you. Now, what is power? Power is nothing, but force multiplied by velocity. So, tau which is shear stress that is acting on an area which is equal to W times L, where W is the width of the belt and L is the length of the of this. So, this tau times the area this is essentially giving me the force and the force multiplied by the velocity of the belt would give me the power. So, force times meter per second is work done per unit time which is nothing, but power.

So, when you calculate that it is 1.73 watts for of power that is needed to make the belt move with a certain velocity. But once again the belt, the power that we calculate is the power to overcome viscous losses to overcome the to account for the work to be done against viscous forces, viscous resistance offered by the liquid. It does not consider any other losses that you can expect that you are going to have in terms of the machineries which are present in such a system. So, a simple use of Navier Stokes equation would give you insights and would let you calculate important parameters for a system which is used to take care of pollution due to spillage of oil in river water.



The next one we are going to discuss about a tangential annular flow of a Newtonian fluid in between two cylinders, one of which is rotating the other is being kept stationary. So, this is a quite common occurrence in many important instruments specially for measuring viscosity. So, what is done is that one cylinder which is let us say this is the outer cylinder and there is an inner cylinder like this. So, the outer cylinder rotates the inner is kept stationary. Now, and there is a liquid in between the two in between the two cylinders.

Now, when the outer one starts to rotate the liquid which is very close to the outer cylinder will also try to move along with it. The motion of the fluid layer close to the outer cylinder will get will get transported more towards the centre because of viscosity. So, one layer of fluid flowing past another layer at the inside would try to move the inside layer as well and so, as time progresses the outer moving the moving outer cylinder will create a flow field in the liquid in between the two cylinders. So, the inner cylinder which was stationary will also try to fill a pool for you to rotate in the direction of the outer cylinder. When steady state has reached then you the if you do not do anything then the inner cylinder will also start rotating though with a lower velocity than the outer cylinder.

Now, what you are going to do is you are going to apply a torque a force a torque on the inner cylinder to prevent any motion of the inner cylinder due to the motion of the outer cylinder and due to the transport of that viscous momentum on to the inner cylinder. So, what is that torque that you must apply from outside to keep it stationary? It is quite easy to measure the torque. So, measuring the torque would give you some idea about what is going to be the viscosity of the liquid in between and this kind of viscometers which rely on the torque to be applied to keep the inner cylinder in this case stationary that they are very good models for certain type of viscometers that try that that that that try to figure out that try to measure what is the viscosity of moderately high viscous materials. So, we need to obtain a formula which would correlate which would relate the torque with the viscosity and that is the whole purpose of this problem a practical situation in which you measured the torque you know the dimensions of the 2 cylinders you know what is the angular velocity of the outer cylinder. So, omega naught is known to you r the radius is known to your k r is known to you the only unknown is the mu of the liquid the liquid in between.

So, is there a way to connect the torque with viscosity? So, that is the purpose of the problem that is why we try to solve the problem to obtain the unknown value of viscosity measure torque and you get the you get the viscosity. So, there is a motion now if you look in this figure the motion is in the in the theta direction, there is no movement in the r direction there is no movement in the z direction the movement is only in the theta direction. So, if that is the case and if I must choose the Navier Stokes the right component of Navier Stokes equation, I am going to choose the theta component of Navier Stokes equation. So, this is how the problem is different from the 2 problems which we have solved before. So, we are going to write the theta component of the Navier Stokes equation and cancel the terms which are not relevant and then arrive at the governing equation.

Solve with no slip condition at the two walls of the rotating cylinders at r = R and r = kR

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{v_z}\frac{\partial v_\theta}{\partial z} + \cancel{\frac{v_r v_\theta}{r}}\right) = -\frac{1}{r}\cancel{\frac{\partial p}{\partial \theta}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \theta}}\right] + \cancel{\rho g_\theta}$$

So, what is this is the theta component of the Navier Stokes equation. Now, if you see and if you try to get rid of the terms which are not going to be relevant here the first thing it is at steady state. So, the first thing is going to be 0 and after that there is no r component of velocity, v r is going to be 0. So, this term would also disappear. The next is due to angular symmetry v theta the nonzero component of velocity is not going to be a function of theta.

So, everywhere it is angular symmetry. So, if you take if you fix the r the v theta is not going to be a function of theta it is same at every point. So, for that case the v del v theta del theta would also be equal to 0. There is no velocity in the z direction. So, v z will also be equal to 0 and since my v r is going to be 0.

So, this term would also be equal to 0. So, the entire convective contribution to momentum transfer for this system which is represented by the 4 terms on the left-hand side and the special term which represents the time varying nature of flow all of those are going to be equal to 0. So, the entire momentum transport in this situation is governed by the viscous transport of momentum or molecular transport of momentum not convective transport of momentum. The next one is that there is no imposed pressure gradient in this system. Since there is no imposed pressure gradient in the system this part would also be equal to 0. When you come to the next term you can see that v theta you can realize that v theta is a function of r or near the near the outer wall near the outer cylinder the velocity of the fluid will be much more than near the stationary inner cylinder where the velocity will close will be close to 0.

So, I cannot do anything with the first term inside the third bracket, but if I think about the second term v theta is not a function of z, v theta is not a function of theta. So, both the terms inside the third bracket would be 0 since v theta is independent of theta and z. And similarly, there is no v r component present here. So, this would also be 0 and there is no gravity force because these are two are two vertically vertical cylinders one inside the others. So, there is no variation, there is no effect of gravity in the theta direction.

$$\mu\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r\,v_\theta)\right)\right]=0$$

So, with these understanding incorporate in resulting in the cancellation of terms which are unimportant in the governing equation there is going to be only one term remaining in the Navier Stokes equation. This once again underscores the beauty of Navier Stokes equation just by thinking what is happening inside and choosing the right component of Navier Stokes equation and cancelling the terms would give you the governing equation without going through any cumbersome shell momentum balance approach. So, this now can this now can be integrated and it is not a partial differential equation because v theta is a function of r only. So, this ordinary differential equation can be solved by the application of no slip conditions at small r equals capital R. That means, on the outer cylinder which is rotating and small r equals k r the inner cylinder which is kept stationary.

$$0=\mu\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d(r\,v_\theta)}{dr}\right)\right]$$

$$v_\theta=\frac{C_1}{2}r+\frac{C_2}{r}$$

So, we are going to, we can we can now then solve this equation and with the first if the integration would give me these two these two integration constants C 1 and C 2. The boundary conditions are at r equals k r the v theta would be 0. That means, the inner cylinder at the walls of the inner cylinder there is no motion and at the outer cylinder the v theta is equal to omega naught r where omega naught is the angular velocity. So, with the incorporation of these boundary conditions to evaluate C 1 and C 2, the expression for v theta can be obtained. And if you look at the expression for v theta it is a function of omega naught r k r etc.

Boundary Conditions: At $r = kR$, $v_\theta = 0$;

$$\text{At } r = R,\ v_\theta = \Omega_0 R$$

$$v_\theta = \frac{\Omega_0 R\left(\frac{kR}{r} - \frac{r}{kR}\right)}{k - \frac{1}{k}}$$

Once again I would request you to solve the problem from the governing equation, apply the boundary conditions do it on your own to check whether or not you are getting the same expression for v theta. This would be a very good practice for you which would come which would which would really be useful while solving problems during the exam. So, this practice is necessary, but once again coming back to the problem I have now an expression of v theta that I can that I can work with. So, what am I going to do next? Since I have the velocity, the velocity distribution I should be able to obtain what is the shear stress. So, the shear stress for cylindrical coordinate system is slightly more complicated than the one that we can use for Cartesian coordinate system that is it is simply not mu times velocity gradient.

$$\tau_{r\theta} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

So, there is a table in the text with the table would be supplied to you in case you require that during the exam, but this is simply mu times r del del r of v theta by r and there is going to be another term in here. So, this shear stress expression would be available to you let me be very clear on that. So, once I have that expression once I choose that expression from the table for tau r theta and why it is r theta you can see it is the theta component of momentum getting transported in the r direction. So, theta component of momentum getting transported in the r direction. So, according to our sign convention it is tau r theta.

So, once I write that then I must see is there any further simplification that I can make? Obviously, if you look at the second term there is no v r there is no velocity in the r direction. So, therefore, the second term in the shear stress distribution which one can obtain from the table, you can simply write tau r theta equals this and the expression of v theta which we have obtained from our previous analysis I can substitute in here and obtain what is going to be the expression for tau. So, this one is the expression for v theta and then you do this d r of this term and if it is on the inner cylinder the or the outer cylinder, the value the expression would contain either r or k r. So, tau r theta will be simply equals this twice mu times omega naught r square and so on. One point here is that you can you can you can appreciate how the shear stress is a function of mu, is a function of rotational speed, is a function of geometry which is represented by the value of r and by the value of k.

$$\tau_{r\theta} = -\mu\left[r\frac{d}{dr}\left(\frac{\Omega_0 R\left(\frac{kR}{r} - \frac{r}{kR}\right)}{r\left(k - \frac{1}{k}\right)}\right)\right]$$

$$\tau_{r\theta} = -2\mu\Omega_0 R^2\left(\frac{1}{r^2}\right)\left(\frac{k^2}{1-k^2}\right)$$

Now, this tau r theta is valid for any r, but we would like to we would like to find out what is the torque required for the to move the outer shaft. So, you to move the outer shaft that means, in that for the case of the outer cylinder r is equal to capital R from the geometry. So, the expression for tau r theta that we have obtained in the previous slide which is this, this is now to be evaluated at small r equals capital R and one more thing this tau r theta the expression of tau r theta is the stress on the fluid. So, the fluid is going to exert something on the cylinder as well which would be equal and opposite and that is why I have this minus tau r theta in the expression. So, after I plug in the expression for tau r theta evaluated evaluate it at small r equals capital R, this is my final expression which connects torque with the length of the 2 cylinders represented by L the viscosity mu the dimensions of the 2 cylinders expressed by r and k and the rotational speed of the outer cylinder.

$$T = 2\pi RL(-\tau_{r\theta})_{r=R} \cdot R$$

$$T = 4\pi\mu LR^2\Omega_0\left(\frac{k^2}{1-k^2}\right)$$

The rotational speed you set in your experiment; you know what is L you measure torque. So, this gives you a direct relationship between the measured value of torque and the unknown value of mu. The task which we start which we have started while solving this problem. So, a simple Navier Stokes Analysis would not only give you the velocity distribution, it would give you the shear stress distribution from which you can calculate the torque at a specific point which would correlate the torque with the viscosity. So, this is a very good model for friction bearings where there is a thin film of lubricant in between 2 moving surfaces and as a viscometer as an instrument which measures the viscosity, quick hat stick viscometer this is a perfect model for this kind of viscometers.

So, direct use industrial and apply application use for Navier Stokes equation and however, we must keep in mind that this is valid for laminar flow. Now, mostly the gap in between the 2 cylinders is kept at a minimum at a small value. So, the these are used for highly viscous fluids and viscosity brings order to the system and therefore, you would expect it to be laminar flow. Now, if the if the in this problem the outer one is rotating the inner one is stationary, that is the that is the usual choice. If the inner one is rotated and the outer one is station kept stationary then the centrifugal force would probably force the liquid to move in the r direction as well.

Here when the outer one is moving the outer layers are moving at a faster velocity as compared to the inner layer. So, there is a definite potential jump that needs to be accomplished for the particle to move in the r direction. Whereas, if the inner one is rotating then the out then the fluid is going to expect it to jump from the inside to the earth outside therefore, creating a velocity in the r direction and if that happens then this analysis would not be valid because we have assumed v r to be 0 in order to obtain the velocity expression. So, in this kind of viscometers the outer one is always rotated the inner one is kept stationary and the expression that we have derived is used to correlate viscosity with torque, the unknown viscosity with the measured value of torque. So, that is all for today's class and we will continue solving similar problems of industrial importance in our future classes. Thank you.