Momentum Transfer in Fluids
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Lecture-17

So, good morning once again. We are going to continue with our analysis of Navier-Stokes equation which we have derived heuristically, not step by step in the previous class. The Navier-Stokes equation apparently looks quite complicated, but in this class, I am going to show you how easy it is to use Navier-Stokes equation for specific problems and how can we derive or how can we arrive at the governing equation for a given problem just following simple logic and by choosing the right component of Navier-Stokes equation which would be pertinent to the case in hand. So, first of all this is just a quick recap of what we have done in the previous class. We defined three derivatives one is partial time derivative where the coordinate system is fixed in space. So, x, y, z is constant.

$$\text{Partial time derivative: } \frac{\partial c}{\partial t} \text{ (x,y,z constant)}$$

$$\text{Total time derivative: } \frac{dc}{dt}, \frac{dc}{dt} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial c}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial c}{\partial z}\frac{\partial z}{\partial t}$$

$$\text{Substantial time derivative: } \frac{Dc}{Dt}, \frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x\frac{\partial c}{\partial x} + v_y\frac{\partial c}{\partial y} + v_z\frac{\partial c}{\partial z}$$

The second one is where the coordinate system has a velocity of its own. So, the dx dt etcetera those terms that is a total time derivative where the dx dt dy dt etcetera they demonstrate or they represent the velocity of the coordinate system. And the third and the most important one for analysing Navier-Stokes equation is the substantial time derivative or derivative following the motion. In this specific case the coordinate system has a velocity equal to the velocity of the fluid at that point.

So, its derivative following the motion or it is also known as substantial time derivative. Based on this and our based on our discussion and analysis we have obtained this form of the equation of motion. So, if you look at the left-hand side its rho times acceleration. So, its rho is mass per unit volume times acceleration. So, essentially this is the force acting on the control volume.

$$\rho\frac{Dv}{Dt} = -(\nabla p) - (\nabla.\tau) + \rho g$$

And then on the second on the right-hand side the first term its del p. So, its pressure force per unit volume and then we have the shear stress per shear force per unit volume and the last one is gravitational force per unit volume. So, what we have done in this case is we have written the Newton's second law of motion for an open system where fluid is allowed to come in and leave carrying with it some momentum associated with the velocity at that point. So, this is nothing but a force balance equation. One more point to note here is that all terms in Navier Stokes equation are expressed as force per unit volume.

So, this has to be kept in mind. This is important because when additional forces are important in specific situations whenever you would like to add those additional forces into this equation to make it applicable for specific applications, we have to be careful about the units of the term that you are including in Navier Stokes equation. So, the from the Navier Stokes equation what you see at the top from the equation of motion that you see at the top if we use a constant rho constant mu and its Newtonian fluid then what we get is known as the commonly known as the Navier Stokes equation. Now, if viscous forces are not present for an inviscid fluid the second term on the right hand side would disappear and what we get is Euler's equation and from Euler's equation one would be able to obtain the Bernoulli's equation which we will see in the second part of this course. So, the Navier Stokes and the Euler equation are quite well known in and their applications in fluid mechanics are many.

$$\rho \frac{\mathrm{Dv}}{\mathrm{Dt}} = -(\nabla \mathrm{p}) - \mu \nabla^2 \mathrm{u} + \rho \mathrm{g}$$

$$\rho \frac{\mathrm{Dv}}{\mathrm{Dt}} = -(\nabla \mathrm{p}) + \rho \mathrm{g}$$

So, what we would do in today's class is to write or to see what are the forms of Navier Stokes equation for rectangular systems, Cartesian coordinate systems, for cylindrical coordinate systems and for spherical coordinate systems. So, for rectangular Cartesian coordinate systems there could be two or three components of velocity v x, v y and v z. So, for each of these components the Navier Stokes equation is provided in the textbook and similarly for the cylindrical coordinate system and for the spherical coordinate system. The equations look quite complicated and I will never expect that anyone would remember those equations. So, for any situation any problems if the use of Navier Stokes equation in any of the coordinate systems is required that would be provided to you.

$$[\partial \rho / \partial t + (\nabla \cdot \rho \mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0$$

So, that is any relation correlation etcetera that would be provided in the question itself. So, you do not have to remember anything in this course. So, let us look at the first the equation of continuity which I said previously that it is the equation of conservation of mass and here you would see that the in the Cartesian coordinate system that is the common form of the of the continuity equation. Cylindrical coordinate systems because of the change in geometry the equation has been transformed and you have this v r, v theta and v z three components which are relevant in a cylindrical coordinate system. For spherical coordinate system it is more slightly it looks slightly more complicated because you have now three components are v r, v theta and v phi.

But the solution of or use of the use of the continuity equation helps us a lot in many problems to obtain the approximate form of the equation, approximate form of the expression for the velocity in a specific direction which I will show you subsequently. But this is what equation of continuity is once again a statement of the conservation of mass. Next comes the equation of next comes the equation of motion for a Newtonian fluid with constant rho and mu or in other words the Navier Stokes equation. So, if you look at the look at the terms over here now the let us talk about v x that is the velocity in the x direction. So, I am writing the x component of the Navier Stokes equation.

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}]$$

Cartesian coordinates (x, y, z):

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

So, if I start with this the first term over here that you see is the transient term. So, it denotes the time rate of change of velocity in the x direction. The second, third and the fourth term they have velocity explicitly appearing in their expression. Now, since the velocity is associated with the moving fluid. So, all these three terms on the left-hand side collectively they are called the convective momentum transport.

The terms which signify the convective transport of momentum in any system. So, if you come to the right-hand side this is the externally imposed pressure in the fluid in the fluid flow field. So, it could be provided by a pump or some such make some such equipment. The terms in the third bracket over all these terms they have this mu in front of them and remember this is a Newtonian fluid. So, this does not have velocity, but it has the velocity gradient associated with it.

So, these are the conductive momentum transport or viscous transport of momentum or molecular transport of momentum. All three have the same, I mean this these terms are represented or expressed in terms of molecular transport or viscous transport and so on. So, they do not require the, they transport the momentum in a direction perpendicular to the flow of the fluid because of the presence of viscosity. So, we have the convective transport of momentum a term which refers to the transient condition and the last term is as you could see is could be the body force which in many situations is essentially the gravity force. So, the complete Navier Stokes equation once again has the convective transport on the left-hand side, the diffusive transport on the right-hand side, you also have the pressure gradient being applied on the flow field and you have the body force.

So, one has to choose which component of which component of the momentum transport which component of the Navier Stokes equation is to be used for a specific application. In the way to choose the right component of Navier Stokes equation be it in the x momentum or x component y component or z component would essentially depend on in which direction you have the flow. If you have flow along the x direction then obviously, the x component of equation of motion or if x component of Navier Stokes equation is to be chosen. And similarly for y and the z component with application with specific application I will show you how this is done. If you go to the cylindrical coordinate Navier Stokes equation, they look slightly more complicated.

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

So, this is the r component of the this is the r component of the Navier Stokes equation in the cylindrical coordinate, this is the theta component and this is the z component. So, if you have a pipe through which a liquid is flowing then this is your r component, this could be your z component and this is your theta component. So, depending on what is the exact situation that you are trying to express that you are trying to model or which component of velocity that you would like to evaluate or express in terms of the properties, in terms of the geometry, in terms of the impulse condition you have to choose the specific component of Navier Stokes equation in cylindrical coordinate system. As you can as you can well imagine that the situation becomes more complicated or rather the terms become more complicated when you go to spherical coordinate system. So, you have r theta and phi and as you could see that the first one is the r component of the of the Navier Stokes equation, the next one is the theta component and then is the phi component of velocity in spherical coordinate system.

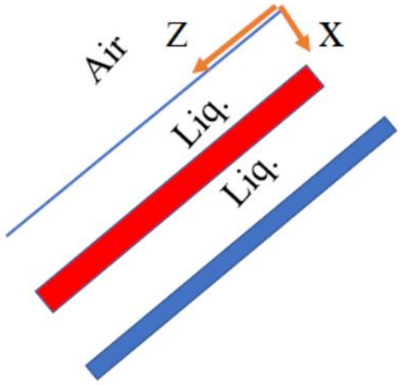
Spherical coordinates (r, θ, ϕ):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\theta$$

Once again, the choice of the of a specific equivalent component or specific component of Navier Stokes equation would depend on what is the principal direction of motion of the fluid and then you choose it accordingly. So, what I am going to do now is I am going to resolve some of the problems that we have done using the shell momentum balance, but this time we are going to use Navier Stokes equation and I can assure you that the end of this class you know how easy it is to use Navier Stokes equation and you will never back to shell never go back to shell momentum balance again. So, let us start with the first problem that is that we have solved in this is flow along an inclined plane, but this time we are going to use Navier Stokes equation. So, this was the geometry which we have solved in that problem in which case in which there was some angle theta and the this was the this is the x component and the principal direction of motion is in the z direction. So, what we have done before is we have chosen a shell like this and we made the balance of the convective flow of momentum, the diffusive or conductive flow of momentum, presence or absence of any pressure gradient in the system and of course, there would be some component of g which is the gravity body force which is acting on this shell on the on the fluid which is contained within this shell.



$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_x}\frac{\partial v_z}{\partial x} + \cancel{v_y}\frac{\partial v_z}{\partial y} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right) = -\cancel{\frac{\partial P}{\partial z}} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z$$

So, this time we are going to do it using Navier Stokes equation. So, if I ask you that which this is a Cartesian coordinate system. So, we will have to go back and use one of these one of these one of these equations. So, which component am I going to use? Now, if you think back in the problem that we I have just shows you the motion is in the z direction whereas, the momentum gets transported in the y in the x direction and it is really wide in the y direction. So, nothing in nothing gets affected in the y direction, but the principal motion is in the z direction.

SS; $V_x = 0$, $V_y = 0$; V_z is not a function of z; No applied pressure gradient; V_z is not a function of y; V_z is not a function of z

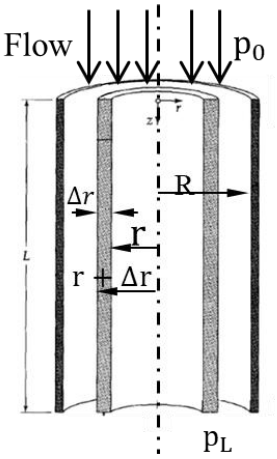
So, therefore, I should choose the z component I should write the z component of equation of motion for flow along an inclined plane and then try to solve it. So, this is what the this is the z component of the equation of motion and then we are going to cancel out the terms and see if we can arrive at the same governing equation as before. So, the first thing that it is a steady state problem. Since it is steady state and now if you look at the equation the first term on the left-hand side is essentially the denotes the transient nature of the flow. So, since it is steady state this term would disappear, there would be no effect of transient in this specific case.

Next comes that we know that there is no velocity in the x direction or in the y direction. The flow is it is an one dimensional flow. So, the flow is only in the z direction. Therefore, both v x and v y are 0. So, the implication of that understanding is that this term v x would be 0, v y would be 0 as well. So, we are slowly therefore, we are slowly discarding or eliminating terms from Navier Stokes equation which are not going to be of any relevance to solve this specific problem.

The third one if you recollect, we said that it is a freely falling film, it has reached steady state and v z is not a function of z. v z here is a function of x, but it is definitely not a function of z not a function of y as well. So, since v z is not a function of z this term would disappear del v z del term would disappear. And similarly, we also said that this is a freely falling film and there is no applied pressure gradient. So, if there is no applied pressure gradient then the first term on the right-hand side would also disappear.

The last point that we have is v z is not a function of y, the plate is wide and therefore, in nothing happens in the y direction. So, if it is not a function of y then this term would also disappear and once again v z is not a function of z. So, the third term inside the bracket in the ins that would also not be present in the final form of the equation. Now, once I discard all these terms and I realize that v z is a function only of x and not of anything else not of t or y or z then there is no need to use the partial sign anymore. So, therefore, my governing equation would simply be the ones the two terms which are left where g z is replaced by the component of gravity in the direction as shown in the figure.

And if you now go back and check with your previous slides previous class notes you would see that this is the same governing equation which we have obtained for this case, but after doing a time consuming and at sometimes cumbersome procedure. So, but in here just writing it that choosing the right component of Navier Stokes equation cancelling the terms would give you the final governing equation in a straightforward manner. So, this is what I wanted to this is I would like to emphasize as the beauty of using Navier Stokes equation for solving problems. Two more examples before I close today. The next one is the second problem which we have done in this class where flow through a pipe using Navier Stokes equation.



$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right\}$$

So, if you look at the figure over here then there is a that is there is a pipe the it is vertically oriented the pressure at the upstream side the pressure at the upstream side is p not, which is more than p l and you have gravity acting in this direction. So, therefore, the flow in this case is assisted both by gravity and by the pressure gradient. So, the pressure gradient is negative that means, as we move in the z direction, this is the z direction as we move in the z direction the pressure decreases and therefore, the flow the flow I mean the pressure driven flow would be from top to the bottom. In order to solve this specific problem, we have assumed a shell which is shown over here. So, this is the shell that we have obtained of thickness delta r which is situated at a distance of r from r from the centre.

So, this is r and r plus delta r which are clearly marked in the figure and across this shell we have done our shell momentum balance. So, the annular area that you see at the top the fluid is going to enter carrying some convective momentum with it and it is going to leave at the bottom from here and carrying the same momentum same convective momentum because the non-zero component of velocity in here is only v z. And v z you can you can you can clearly see that because of viscosity and because of the interaction of the fluid with the solid wall the velocity is going to be vary with r. So, v z is a function of r v z is not a function of z the axial location, it is a fully developed flow and therefore, and I will talk about fully developed flow in a while, but this essentially tells us that the velocity is not a function of z and because of angular symmetry v z is also not a function of theta. So, if you specify your r location then at that same r location at different values of theta the velocity would be identical.

Steady State, $V_r = 0$, $V_\theta = 0$, V_z is not a function of θ, V_z is not a function of z, Presence of an applied pressure gradient

$$0 = \rho g_z - \frac{dp}{dz} + \mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

So, therefore, v z is not a function of theta as well. And therefore, we must know, we must choose the right component of Navier Stokes equation in cylindrical coordinate system. It is it is it must be quite clear to you know that since the motion is in the z direction. So, I am going to choose the z component of the Navier Stokes equation in cylindrical coordinate system which looks like what that you see in that at the top of the slide. And then we are going to again apply our understanding of which term would finally, remain in the governing equation and which can be neglected.

So, the first thing is steady state and we know that if it is steady state the first term on the left-hand side would have to disappear. Second is that v r is 0 there is no velocity in the r direction all velocities are in the z direction. So, if v r is 0 this term would be equal to 0. Then the second is v theta is 0 there is no velocity in this direction. So, if v theta is 0 this term would also disappear, v z angular symmetry not a function of theta.

So, if that is the case then that term the second term inside the second bracket would also be equal to 0. And v z is not a function of z. So, you have the fourth term on the left-hand side and the last term on the right-hand side both would be equal to 0. And lastly, we have a pressure gradient present in the system. So, unlike the previous problem I cannot say that d p del p del z is going to be 0.

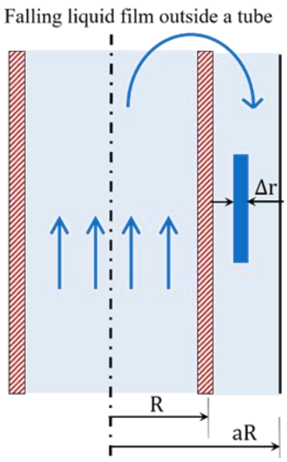
So, what I have in this after I apply all of them is that there are three terms which are remaining on the on the right-hand side. And once again the velocity v z the component of velocity v z is a function only of only of r and not of theta and not of z. So, therefore, the partial sign can now be dropped and what we get is an ODE ordinary differential equation as my governing equation. And in this case g suffix z, the z component of gravity would simply be equal to g. And once again the same equation the same governing equation we have obtained the same equation for the by doing a shell balance over this, but here it is just a one step one step method.

So, you need to choose the component and cancel the terms that are that have no relevance in for a specific problem. So, this is how we proceed with the use of Navier Stokes equation. But there is something about fully developed flow that I think I should clarify which I have mentioned in this. So, what is fully developed flow? For a fully developed flow you have a pipe or it could be two parallel plates and flow is entering at this point this being the centre line. Initially when you reach the fully developed condition this is the z direction and let us call this is at the r direction.

Initially the v z is going to be a function of r and it is going to be a function of z. So, the profile would look like something like this. In this region the flow is going to be affected by the presence of viscosity by the presence of the interaction of the fluid with the solid wall. As you move into higher and higher z this zone will increase in size and it will look something like this. And we know that when it reaches the fully developed condition it is simply going to be a parabolic distribution.

So, this region where the velocity is going to be a function of both r and z means that the flow is developing, it is a developing flow. But once it reaches the fully developed condition the velocity is going to going to take the normal parabolic form and we call it as fully developed flow. So, in up to this point v z is a function of both r and z, but over here v z is going to be a function of r only and this is what we call as fully developed flow. So, in fully developed flow the velocity is a function of in this specific case of r only not on z. And of course, the length of the length over which the flow changes from flow are going to be developing and then becomes fully developed it is known as the entrance length.

And the entrance length as you can see it would depend on the properties of the fluid, it would depend on what is the velocity with which the fluid is fluid is entering and the geometry of it. But keep in mind that fully developed flow is a case in which it is a function only of r and not of z. Now, we move on to the last problem that I have solved and I am going to resolve it using Navier Stokes equation. So, if you recall that we have some flow that is going up and then it reverses its direction and starts falling on the outer wall of the pipe with a constant velocity and we are going to analyse the region after which the flow is fully developed. That means, in here the only nonzero component is simply going to be equal to v z and v z is not going to be a function of z and we understand that v z is going to be a function of r.



$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right\}$$

So, we have done the shell momentum balance for this specific case. So, once again the relevant component here would be the z component because that is the direction in which we have motion. And then steady state first term disappears v r there is no radial velocity component. So, v r would be 0 there is no component of velocity in the theta direction.

So, this part would be 0, v z is not a function of theta. So, that term would disappear it is fully developed v z is not a function of z. So, you do not have these two terms and there is no pressure gradient also, it is a freely falling liquid. So, liquid film. So, the d p d z term del p del z term would also disappear. So, what we end up with is an ordinary differential equation that relates the velocity, the falling velocity of the film in the x direction with the physical property viscosity and the nonzero body force present here which is gravity.

And then once you have the governing equation then you can use those boundary conditions, relevant boundary conditions of no slip at this location and no shear at this point exactly the same way as we have done before. So, with these three examples I hope I am able to show you how easy and how convenient it is to use Navier Stokes equation for solving problems of fluid motion, viscous motion of fluids. We will continue solving or trying different problems using Navier Stokes equation and see how to obtain the velocity profile, the shear stress profile, the force and so on in our subsequent lectures. Thank you.