Momentum Transfer in Fluids
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Welcome to this session of Momentum Transfer in Fluids. We were discussing various deformations that a fluid particle undergoes during the motion of a fluid. It comes under the general umbrella of inviscid flow, elements of inviscid flow. So, what we discussed at the end of last class was that we have a fluid particle that may undergo translation, rotation, angular deformation and linear deformation. So, these are the pictures that we discussed at the end of last class.

Fluid Translation: Acceleration of a Fluid Particle in a Velocity Field

$$\vec{a} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{Local acceleration}} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{Convective acceleration}} = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{Total acceleration}}$$

In Rectangular coordinate system

$$a_{x_p} = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t} = \frac{Du}{Dt}$$

$$a_{y_p} = \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w + \frac{\partial v}{\partial t} = \frac{Dv}{Dt}$$

$$a_{z_p} = \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w + \frac{\partial w}{\partial t} = \frac{Dw}{Dt}$$

Now, if we try to see what is this fluid translation, translation primarily involves the acceleration that we discussed just in the previous session. We have to talk about the local acceleration and convective acceleration that will give the total acceleration which is expressed as a substantial derivative. So, in rectangular coordinate system, we have you can see the pattern here. We have the pattern here that here when it comes to the x direction, I see that u is repeating everywhere.

These are the u's repeating and u, v and w they are appearing here. When it comes to the y direction, the v is repeating everywhere. Similarly, in case of z, w is repeating. So, this you may see a pattern here and that you need to appreciate and you need to keep in mind. So, this is what is acceleration in a rectangular coordinate system.

In Cylindrical coordinate system

$$a_{r_p} = V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{{V_\theta}^2}{r} + V_z \frac{\partial V_r}{\partial z} + \frac{\partial V_r}{\partial t}$$

$$a_{\theta_p} = V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z \frac{\partial V_\theta}{\partial z} + \frac{\partial V_\theta}{\partial t}$$

$$a_{z_p} = V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial t}$$

Now, we had talked about the cylindrical coordinate system. You may recall that instead of x, y and z, instead of expressing a point differential volume in Cartesian system by x, y, z, we could write this as r and this angle as theta and write x is r cos theta, y is r sin theta, z remains same as it is and we can construct a differential element whose one side is the arc length r d theta and this would be dr and delta z will remain same. So, we can do all these balances on the cylindrical coordinate system on a differential element and if one does that then the expression for acceleration takes the form as it is given here. You can see here the vr when it comes to this acceleration in the r hat direction and this is the acceleration in the theta hat direction, r hat is this is the r direction, this is the theta direction. So, r hat is the unit vector in r direction, theta hat is the unit vector in theta direction and you have the z axis also.

So, this is the you have the this is k hat or standard k hat. So, these here you can see that it is not straight it is vr del vr del r is fine, but here for del vr del theta you have v theta by r and then you have an additional term minus v theta square by r and then you have vz del vr del z plus del vr del t. So, these you can see certain here the trend change it is vr v theta divided by r. So, if one wants to work with cylindrical coordinate system and try one tries to find out what is the acceleration. So, these are the expressions one has to follow.

We do not want to get into the derivation of it, but just be aware that we are working on Cartesian system, but alongside the cylindrical coordinates the the working of cylindrical coordinate system is also very much there. Here I go through a quick example of why this how this substantial derivative comes into play. Think of a flow through a converging channel that I mentioned earlier. So, let us say the velocity along the center line is given by this expression. Velocity exists only in the x direction and that is that is that is this direction this is the x direction.

Consider two-dimensional, steady, incompressible flow through the plane converging channel shown. The velocity on the horizontal centerline is given by $\vec{V} = V_1\left(1+\frac{2x}{L}\right)\hat{\imath}$. Find an expression for the acceleration of a particle moving along the centreline using (a) the Eulerian approach and (b) the Lagrangian approach. Evaluate the acceleration when the particle is at the beginning and at the end of the channel.

For a steady, two-dimensional, incompressible flow through a converging channel. The velocity field is given by

$$\vec{V} = V_1\left(1+\frac{2x}{L}\right)\hat{\imath}$$

On x-axis along centerline of the channel, the acceleration of a particle moving along the centerline using the Eulerian approach, and x-component of acceleration,

$$a_{x_p}(x,y,z,t) = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

On the axis, $v = w = 0$ and $u = V_1\left(1+\frac{2x}{L}\right)$, so for the steady flow we obtain

$$\Rightarrow a_{x_p} = u\frac{\partial u}{\partial x} = V_1\left(1+\frac{2x}{L}\right)\frac{2V_1}{L}$$

Therefore, the magnitude of acceleration at the entry (x=0) and at the exit (x=L) are $\frac{2V_1^2}{L}$ and $\frac{6V_1^2}{L}$.

$$\Rightarrow a_{x_p} = u\frac{\partial u}{\partial x} = \frac{2V_1^2}{L}\left(1+\frac{2x}{L}\right)$$

$$u_p = \frac{dx_p}{dt} = V_1\left(1+\frac{2x_p}{L}\right)$$

$$\Rightarrow \int_0^{x_p}\frac{dx_p}{\left(1+\frac{2x_p}{L}\right)} = \int_0^t V_1\,dt \quad \Rightarrow \frac{L}{2}\ln\left(1+\frac{2x_p}{L}\right) = V_1 t$$

$$\Rightarrow x_p = \frac{L}{2}\left[e^{\frac{2v_1 t}{L}} - 1\right], \quad \Rightarrow u_p\ (t) = \frac{dx_p}{dt} = V_1 e^{\frac{2v_1 t}{L}}$$

$$\Rightarrow a_{x_p}\ (t) = \frac{du_p}{dt} = \frac{2V_1^2}{L}e^{\frac{2v_1 t}{L}}$$

Times at which the particles are at x=0 , and x=L are

$t(x_p = 0) = 0$ and

$t(x_p = \mathrm{L}) = \frac{L}{2V_1}\ln(3)$

$$\Rightarrow a_{x_p}\ (t=0) = \frac{2V_1^2}{L}$$

$$\Rightarrow a_{x_p}\left(t = \frac{L}{2V_1}\ln(3)\right) = \frac{6V_1^2}{L}$$

So, velocity exists only in the x direction in the y direction typically when you have a flow through a pipe you would have the pressure gradient only in one direction the other direction there is no pressure gradient in the other direction there is because otherwise there will be a cross flow and cross flow there is no reason to believe that there will be a cross flow here. So, the velocity is in entire the velocity is in x direction and he is given at the central line velocity is given as $\vec{V} = V_1\left(1+\frac{2x}{L}\right)\hat{\imath}$. So, let us say this is the velocity. So, you can see that when x is equal to 0 which is the case at the inlet. So, at the inlet x equal to 0 it is v1 simply v1 and at the outlet x equal to let us say this distance is L.

So, at the outlet it is 2 L by L. So, it is 2. So, 2 plus 1 3 v1. So, the velocity at the central line it has increased from V1 to 3 V1. So, that is the acceleration.

So, we can see this, but obviously, there is no time derivative here. So, if you expect that I take a it take the derivative with respect to time and get an acceleration definitely you cannot get this because this is an Eulerian this is a velocity field not velocity of a particle. So, we need to resort to the exercise that we have taken that substantial derivative. So, we have only acceleration in x direction y direction z direction there is no velocity component. So, you have acceleration in the x direction that is $\frac{Du}{Dt}$ only u we are considering because we are interested in the x component of the velocity because y component z component does not exist.

So, that is given by $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$. So, v and w is equal to 0 that is understood and this is and for the steady flow. So, you have v and w is 0 they are gone and $u\frac{\partial u}{\partial x}$. So, if you take del u del x of this if you take derivative of this with respect to x v1 is a constant its derivative is 0 v1 into 2 x by L its derivative with respect to x would be 2 v1 by L. So, u del u del x.

So, $V_1\left(1+\frac{2x}{L}\right)\frac{2V_1}{L}$. So, this is contribution of u. So, this is the u here and del u del x when you take the derivative of this with respect to x. So, this is only v1 into 2 by L. So, that is essentially showing here as $\frac{2V_1}{L}$.

So, this is basically del u del x term and this is the u term. So, you can see here that if someone wants to know the magnitude of this acceleration at the entry point that is at x equal to 0 and at the outlet x equal to at the exit x equal to L then it would be at x equal to 0 this would be v1 into $\frac{2V_1}{L}$. So, it would be 2 v1 square by L. So, this is the case when x equal to 0 and when you put x equal to L here. So, it would be 2 L by L.

So, this becomes 2 2 plus 1 3. So, 3 into 2 6, $\frac{6V_1^2}{L}$.. So, that is exactly what we see here $\frac{6V_1^2}{L}$.. So, these are the acceleration values. So, acceleration exists and we have to resort to Eulerian framework otherwise we will not get the acceleration.

If someone sticks to it that I will find out by counting the particles I want to go by some way we do not want to use this substantial derivative then the other way around would be you have to define first the u the velocity as dx dt and that velocity is v1 into 2 x by L, but this is a particle velocity. So, we are putting a subscript p and then what we do here is dx dt is equal to this. So, we need to do an integration from time 0 to t, this has to be t time 0 to t the distance travelled is 0 to x p. So, dx p divided by this whole thing because this is the function. So, dx p by 1 plus 2 x p by L.

So, that is equal to v1 and this dt goes there. So, v1 dt and then you integrate it. When you do this integration you end up seeing when you do this integration you will see that this is the expression one will get

$$\int_0^{x_p}\frac{dx_p}{\left(1+\frac{2x_p}{L}\right)}=\int_0^t V_1\,dt \quad \Rightarrow \frac{L}{2}\ln\left(1+\frac{2x_p}{L}\right)=V_1 t$$

So, when you take the derivative of this with respect to 1 of 2 x p by L. So, when you take the derivative with respect to this you will get this becomes divided by 2 by L and then that 2 by L when it comes out it would be L by 2 because 2 by L is in the denominator this is a divided sign. So, you have this L by 2 coming in here and this gives you ln of 1 plus 2 x p by L and that is equal to v 1 into t. So, the x p then becomes equal to from here you get 1 plus 2 x p by L would be equal to ln of this. So, it would be e to the power v 1 2 this 2 will go there.

So, this 2 will go there this L will go there. So, 2 v 1 t by L. So, e to the power 2 v 1 t by L or in other words x p would be. So, 2 x p by L would be then the 2 x p from here you get 2 x p by L would be equal to e to the power 2 v 1 t by L minus 1. So, then this x p would be equal to then this 2 by L will go to the right hand side.

So, it would be L by 2 into this quantity that is what you see here if I take this that is exactly what you see here x p is equal to L by 2 into e to the power 2 v 1 t by L minus 1. So, this is coming from there. So, once you have these x p then u p would be derivate. So, what we are trying to do we first of all if we want to get an acceleration we have to express this in terms of the we have to express this in terms of time. So, we have to bring in with respect to time not with space.

So, that has to be brought in. So, you can see now the position of a particle is given in as a function of time and then this now you take a derivative of this with respect to t to get the velocity which is dx p dt. If you take a derivative of this with respect to t then it would be L by 2 remains as it is e to the power 2 v 1 t by L. So, that would be so, dx p dt here it will be L by 2 remains as it is L by 2 into minus 1 that would be constant. So, this derivative would be 0 L by 2 e to the power this. So, it would be d of 2 v 1 t by L e to the power 2 v 1 t by L and then d of 2 v 1 t by L dt.

So, then what you have here is e to the power 2 v 1 t L that remains same. So, that is what we see e to the power 2 v 1 t by L and the derivative of this 2 v 1 t by L with respect to t would be 2 v 1 by L. So, if you multiply with 2 v 1 by L 2 and L will cancel out when this 2 v 1 by L comes out it will cancel out and so, you have only one left with only v 1. So, this becomes the expression for velocity v 1 into 2 v 1 t by L and then when you want to now if you want to do acceleration this you can take directly the derivative of this with respect to time and so, you get this acceleration as 2 if you take it with respect to time. So, again it would be v 1 into you need to do the same thing e to the power 2 v 1 t by L to this and then this derivative.

So, then again from here 1 v 1 by L will come out 2 v 1 by L will come out. So, that 2 v 1 by L into e to the power x that remains same. So, this becomes the acceleration. So, then if we try to find out just the way we had done it acceleration at the entry and acceleration at the exit. So, we have to find out at what time because now it is written in terms of time.

So, at what time the particle was at inlet and at what time particle is at outlet. So, inlet means x equal to 0 and outlet means x equal to L. So,

$$t(x_p = 0) = 0 \text{ and}$$

$$t(x_p = L) = \frac{L}{2V_1}\ln(3)$$

So, you have to go to this expression for x p the x p expression is here this is the xp expression. So, we need to find out what should be the value of t such that this xp is L.

So, xp is equal to L for what time. So, if you do that. So, this becomes L this L and this L will cancel out. So, it would be 2 will go to that side 2 and 2 and this 1 will come to that side this 1 2 plus this 1 3. So, it would be 3 is equal to e to the power 2 v 1 t t this t represents the time at which the particle have reached outlet 2 v 1 t by L.

So, this is the exponential of this. So, in other words you have the time would be equal to you if this is this is. So, then you write this as ln 3 then this exponential will be gone and then time would be equal to $\frac{L}{2V_1}\ln(3)$.

So, now, with these values of t 0 and $\frac{L}{2V_1}\ln(3)$ you put it there and try to find out what is acceleration. Acceleration here is this. So, instead of t if you put t as 0 if t is 0 then this e to the power 0 is 1. So, it will be $\frac{2V_1^2}{L}$ which matches with $\frac{2V_1^2}{L}$ here our Eulerian framework what we got and if you want to find out what is the acceleration at the outlet. So, for that we have to put the t as this t would be then we have to put this t as L by 2 v 1 L by 2 v 1 these are all this v 1 is same as only it has become a lower subscript, but the same v 1 this is the same v 1.

So, L by 2 v 1 is the t and then so 2 v 1 by L L by 2 v 1. So, this becomes simply e to the power 1 because L 2 v 1 2 v 1 L will cancel out. So, it would be e to the power 1. So, e to the power just a minute acceleration would be t is equal to t by 2 v 1 as a t by 2 v 1 and then there is this L n 3 L n 3 is also there. So, these cancel out 2 v 1 and L 2 v 1 by L and L by 2 v 1 cancel out.

So, it becomes e to the power L n 3 and this e to the power L n 3 would be 3. So, 3 into 2 6 6 v 1 square by L. So, that is what we get as 6 v 1 square by L which is same as this. So, you can now probably you will appreciate the Euler's in framework I mean what is the advantage. If you want to pursue if you want to track the particle with respect to time you have to do this exercise to find this simple acceleration and inlet and outlet whereas, if you work with this expression for acceleration the substantial derivative terms if you work with it will be much easier for you to come up with the values of acceleration.

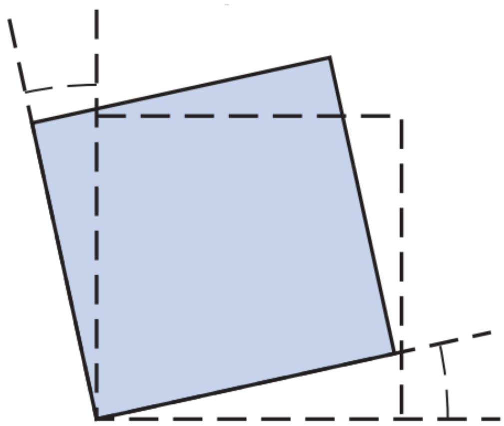
You can see that the amount of calculations involved here. In fact, this is the this framework on the right hand side that we have shown this is primarily a Lagrangian way of handling it here if you work with this framework you can probably continue doing it, but this is not the Eulerian approach. Eulerian approach is what we give in the left hand side and if you work with a velocity field and then you stay happy with $\frac{\partial u}{\partial t}$ and said that and you say that ok there is no acceleration, acceleration is 0 that is not correct. So, with that so, this is as far as the translation is concerned. The next thing that I would be looking at is rotation.

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

ω_x refers to the rotation about x-axis

ω_y refers to the rotation about y-axis

ω_z refers to the rotation about z-axis



Fluid rotation. So, rotation is you it would be around I mean this is let us say a differential element of size let us say Δx and Δy I have this differential element and then these differential element after some time I find that this differential element has rotated around its center line around its center point this has rotated. So, if you it is not exactly this rotation is not exactly around the center point this is something else, but we what we will be discussing is we take up a cross wear and then that around that cross wear if there is a rotation. So, then how do you characterize it and I have I mentioned this earlier and I repeat again that such rotation is possible when you have shear stress coming into play viscosity coming into play. So, this is not exactly the inviscid flow, but I need to understand where my where I am considering inviscid flow and where I am not considering inviscid flow because this rotation as far as rotation is concerned is the viscous flow, we need to we need to look into it and see if we want to make an assumption of inviscid flow what all assumptions what all what all additional assumptions we may have to look into. So, $\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$, t this ω_x is what now that is that is one thing it is refers to the rotation about x axis.

Now, rotation let us say I have this is the fluid element let us say I have a differential element this is the fluid element this is the center line this is this is the center point around which the fluid element is fluid element is rotating around which the fluid element is rotating. Now, you need to understand that if this axis is x this axis is y and that and the z is perpendicular to the screen then any rotation of this plane that would be referred as omega z because this omega itself is a vector and it has is i hat j hat and k hat components and the rotation of x y plane rotation of the fluid element in x y plane that would be counted as omega z because perpendicular to the plane is the z axis. So, this any rotation of this plane would be given by omega z. Similarly, when it comes to omega x it is essentially the rotation of y z plane and you understand what this y z x z or x y planes because we had considered a differential element you may recall and then we have these forces acting from the right face left face etcetera. So, in the differential element we have these faces whose area vectors are having a direction of x y and z.

Motion of a fluid element in x-y plane

y-component of velocity at point 'o' is v_0

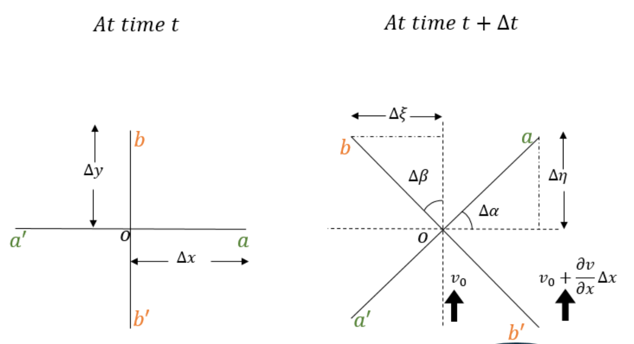
y-component velocity at point 'a' is $v_0 + \frac{\partial v}{\partial x}\Delta x + \cdots$

Angular velocity of line $\overline{oa}$ is

$$\omega_{oa} = \lim_{\Delta t\to 0}\frac{\Delta\alpha}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\eta/\Delta x}{\Delta t}$$

$\Delta\eta$ is the extra movement of 'a', over and above the movement of 'o' = $\left(\frac{\partial v}{\partial x}\Delta x\right)\Delta t$

$$\omega_{oa} = \frac{\partial v}{\partial x}$$



So, now, rotation is here we are talking about the rotation of those faces. So, this fluid rotation, though it is not inviscid flow, we need to understand where what this fluid rotation constitutes to the velocity field, and when we assume that my flow is inviscid, then what condition must I satisfy? So, this is a classical example of the motion of a fluid element in the x-y plane. You can see that the cross wire that I was referring to it is a a' b b'. So, this is the cross wire a a prime b b prime and this dimension is Δx this dimension is Δy.

So, this here you will see that this cross wire gets rotated. So, at time t and at time t +Δ t at time t plus delta t you can see that the cross wire has rotated cross wire has rotated anti clockwise and because of that the a has a is shifted now and similarly b got shifted here. So, now, it is because of this how do we characterize this rotation? First of all if the rotation if there is a rotation this side is Δx this side is Δy and if there is a rotation by an angle $\Delta\alpha$ then you can see here that fluid is in motion. So, the fluid this point is exposed to a velocity let us say vo what is the point o and here this point is exposed to a velocity which is greater than vo which is greater by what amount vo plus del v del x delta x. Again we are taking Taylor series expansion and working only with the first order term assuming this Δx is small and the velocity change is linear.

So, here it is vo and here it is vo plus del v del x delta x that means this is x axis. So, in the x axis the v changes. Now, we are talking about v here not u mind it. So, this is the v, v means velocity in y direction. Here the velocity in y direction will be little different it would be v o plus something and that something is $\frac{\partial v}{\partial x}\Delta x$ I mean we just took Taylor series expansion ignoring higher order terms.

So, in the x direction v is changing from v o to v o plus del v del x maybe further down since this distance is delta x. So, that is what it is. Why we have to consider this? Because any rotation if this cross wire had to rotate then only this point has to rotate some it has to travel by some extra amount. So, over time Δt point o has traveled by some distance it is translation happening right you have acceleration and everything. So, point o is accelerating point this point is also accelerating for this point is also undergoing the similar treatment only thing is there is some special extra velocity it has this point has.

So, what is that extra velocity? because of this extra velocity, when I multiply it by time delta t, I will have some extra displacement, and because of this extra displacement, this point will travel by some extra amount, and that is causing the rotation. So, that is essentially this is $\Delta\eta$. So, what we are trying to find out is over time Δt with this extra velocity. So, that means, this is v naught this is v naught plus. So, this is the extra velocity with this extra velocity if I travel for a duration of delta t how much extra I will travel and that is essentially what is $\Delta\eta$.

So, this is what we will find out and then this $\Delta\alpha$ would be the angular deformation etcetera we will look into that angular velocity we will look into that. And similarly the point B also will move, but it will move in the negative x direction and that value is $\Delta\xi$. So, what we will do is we will combine these and try to find out suppose I get I am given a velocity field some expressions so and so i hat plus so and so j hat plus so and so k hat can I by looking at that velocity field tell whether this there is rotation involved or there is no rotation because rotation involved means flow is not inviscid anymore flow is viscous, but if the rotation is not there then I know for sure that it is an inviscid flow. So, there are some velocity profiles where the rotation will not be there some velocity fields where rotation will be there. So, how do we differentiate these velocities? So, what we do here at this point is I will continue this exercise of motion of fluid particles.

So, what I have done in this module in this class is I mostly try to give you some idea what are various motion of particles that are possible and then in the next class I will discuss again the motion of fluid particles. So, essentially we are looking at probably the next week there would be some other lecture modules posted parallely. So, I will come back after a week with a new chapter altogether which is motion of fluid particles and I have already done the groundwork in terms of translation, rotation, angular deformation, but I need to do a little bit of mathematics to it I mean we need to find out what are the conditions by which we can characterize them. So, we will after next week I will come back with a new chapter on motion of fluid particles where these essentials I will just simply assume that you have already studied and I will build on it and further down I will do some example problems. That is all I have for today. Thank you very much for your attention.