Momentum Transfer in Fluids
Prof. Somenath Ganguly
Department of Chemical Engineering
IIT Kharagpur
Week-03
Lecture-14

Welcome to this session on Momentum Transfer in Fluids. We have been discussing inviscid flow. In particular, we were using Newton's second law and trying to develop some special cases of fluid statics or fluid is being hauled on an accelerating platform and as a third case the fluid flowing through a conduit. So, at the end of the last class, we mentioned that this is the equation we have been discussing, $-\nabla p + \rho\vec{g} = \rho\vec{a}$, that is the expression we have already obtained from by applying Newton's second law and doing a force balance on a differential volume.

So, what we are going to do now is we can extend it as before we said,

$$-\nabla p + \rho\vec{g} = \rho\vec{a}$$

$$-\left(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} + \frac{\partial p}{\partial z}\hat{k}\right) + \rho\left(g_x\hat{\imath} + g_y\hat{\jmath} + g_z\hat{k}\right) = \rho\left(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}\right)$$

So, now, if we are matching x with x, y with y, and z with z, that has to be done, but before that, one must note here that pressure variation in the z direction. So, what is the z direction? We have pressure. So, we are talking about this is the tank which is being hauled at an acceleration a_x only existing a_y, a_z these are 0.

So, you can see $a_y = a_z = 0$, only a_x existing and I said that if the liquid level is this. So, then this is going to be the modified level since the tank is accelerating. So, we have a_x existing, and similarly, I see that this is the direction of x, this is the direction of y, and z is perpendicular to the screen. So, perpendicular to the screen we do not see any reason for this pressure to vary perpendicular to the screen. Pressure can vary.

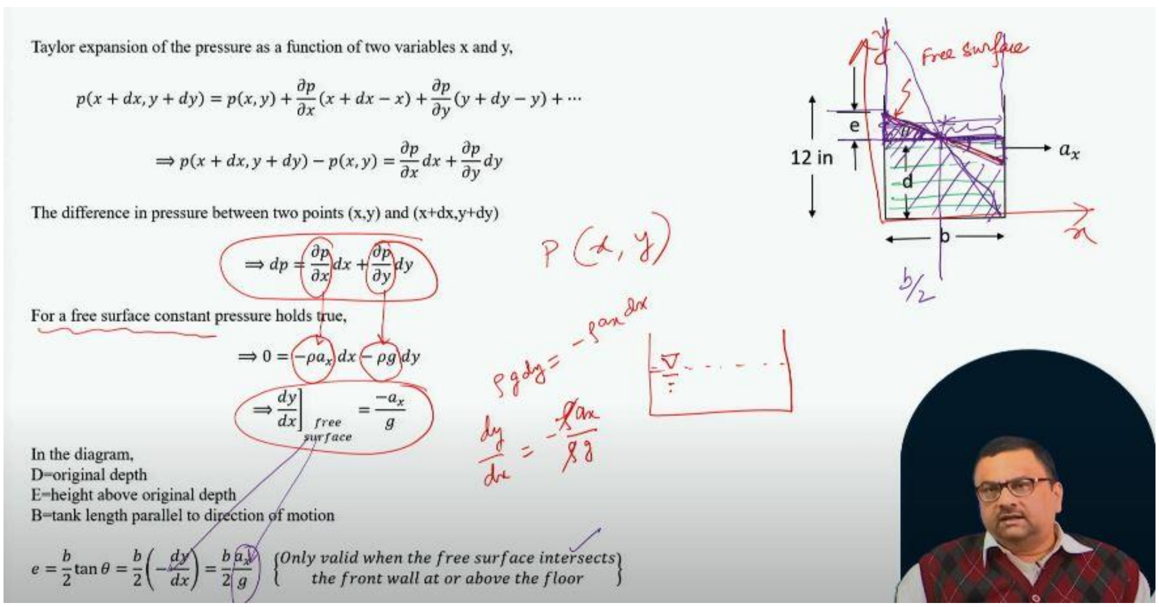
There is no reason for pressure to vary in the z direction, which means the pressure is not a function of z, and $\frac{\partial p}{\partial z} = 0$. That is what it means. Similarly, g_x in x direction gravity does not work, z direction perpendicular to the screen gravity does not work, only gravity works in y direction, but in negative way because gravity is downward. So, that is why we have $g_y = -g$, g is the acceleration due to gravity. So, now if we put these so, we can see here $\frac{\partial p}{\partial z}$ does not exist or g_x and g_z does not exist, and a_y and $a_z$does not exist. So, now, if we pull the remaining terms, this is what we get

$$-\frac{\partial p}{\partial x}\hat{\imath}-\frac{\partial p}{\partial y}\hat{\jmath}+\rho g_y\hat{\jmath}=\rho a_x\hat{\imath}$$

So, if we pull the î and ĵ terms together and all î terms in one place and ĵ terms in another bracket, we see that here $\left(-\frac{\partial p}{\partial x}-\rho a_x\right)\hat{\imath}+\ \left(-\frac{\partial p}{\partial y}-\rho g\right)\hat{\jmath}=0\hat{\imath}+0\hat{\jmath}$

So, since left hand side I have î and ĵ and right hand side I have essentially $0\hat{\imath}+0\hat{\jmath}$. So, I match this with the 0 and this with the 0.

So, we get $-\frac{\partial p}{\partial x}=\rho a_x$, and similarly, this goes to the right-hand side $-\frac{\partial p}{\partial y}=\rho g$. So, this is something which we note that means this is the variation of pressure in x and y direction. In the z direction, I do not have any reason to believe that pressure is a function of z.



What we do next is we take a Taylor series expansion of p and if we only work with the first order terms we see that essentially or generally we can write the difference in pressure between two points pressure dp is given by $dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy$, because we have already agreed that the pressure is a function of x and y only pressure is not a function of z. So, $dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy$. So, this is called the new line that is created and this is the equilibrium line that was there when there was no acceleration. So, now, if we see, I mean, what property does it have, what property does this line have to satisfy? First of all, these lines represent a free surface.

What is a free surface? Free surface, you might have seen that if a lot of drawings, etcetera liquid level is there and then there is a symbol like this. So, this is, so to say, a free surface. So, on a free surface, one condition is that along the free surface, pressure must be constant. Otherwise, the fluid would have moved in another direction, and the free surface could not have been maintained. So, along a free surface the $dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy$ and $\frac{\partial p}{\partial x} = -\rho a_x$, and $\frac{\partial p}{\partial y} = -\rho g$. So, then we can we can write this $\frac{\partial p}{\partial x}$ as $-\rho a_x$ and $\frac{\partial p}{\partial y}$ as $-\rho g$ and this dp would be equal to 0 because of the condition of free surface.

Along the free surface pressure is constant. So, the differential of the pressure is 0. So, in that case if we take one of them to the left hand side and write $\left.\frac{dy}{dx}\right]_{\text{free surface}} = \frac{-a_x}{g}$, because if you take $\rho g dy$ to the other side it would be qual to$-\rho a_x dx$. You will have if you bring in $\frac{dy}{dx}$.

So, $\frac{dy}{dx}$ would be equal to $\frac{-\rho a_x}{\rho g}$. So, rho and rho will cancel out. So, you have $\frac{-a_x}{g}$. So, this gives you the $\frac{dy}{dx}$ this is the free surface. So, this gives me the slope $\frac{dy}{dx}$ essentially, this is my x, and this is my y.

So, $\frac{dy}{dx}$ for the free surface. So, this gives me the slope. So, the slope is negative. That is one thing I intuitively have from our regular observations. We have already seen that the slope is expected to be negative. So, $\frac{dy}{dx}$ is for the free surface is negative and the magnitude is $\frac{-a_x}{g}$. Then what is important for me is how far the liquid level will go up on this side.

So, that has to be figured out. So, to do that one may have to do some kind of volumetric balance. Volumetric balance in the sense we note that initially, the liquid is this and later on, this is the liquid volume, and no volume is added. So, say the liquid and the static state has to be same as the liquid volume that you have in the deformed state. So, that means, what that means is area z is perpendicular to the screen.

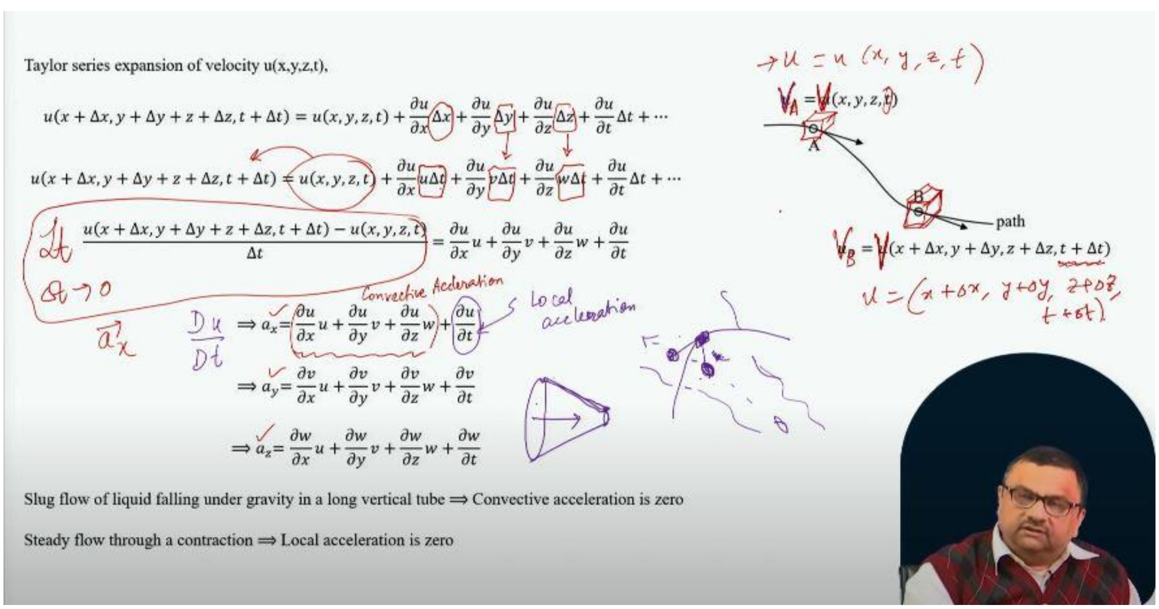
So, these areas in the z direction there is no activity. So, the area of this triangle has to be equal to area of this triangle. That is one condition that these two areas have to be equal, the volumetric balance has to be satisfied. On top of that, these angle is equal to this angle. So, if this angle is equal to this angle, this is 90 degrees. This is 90 degrees. So, this angle is equal to this angle.

So, you can then immediately conclude that this length is equal to this length. So, this length is equal to this length by comparing the two triangles. So, in that case you know for sure that this length is $\frac{b}{2}$. So, once you know this is $\frac{b}{2}$ once you know what this angle θ is $\tan\theta = \left(\frac{dy}{dx}\right)$. So, from there, you know, and on the other hand, $\tan\theta$ would be equal to $\frac{e}{b/2}$.

So, from there, $e = \frac{b}{2}\tan\theta = \frac{b}{2}\left(-\frac{dy}{dx}\right) = \frac{b}{2}\frac{a_x}{g}$

So, from here we can find out what is this extra height the liquid will gain because of this acceleration because e is that extra height. Of course, there is a rider only valid when the free surface intersects the front wall at or above the floor. That means if you are accelerating it too hard, then you would expect the liquid level to go all the way to this, and even if you go further, then the liquid level will go to this side.

So, then this whole calculation would be again you have to redo it. So, that is there, but otherwise, as long as this rider is, I mean, as long as this condition is satisfied, the e that extra height it will gain is given by $\frac{b}{2}\frac{a_x}{g}$. a_x is the acceleration by which you are pulling the truck, b is, I mean we are assuming it to be a rectangular tank. So, this side is b and g is acceleration due to gravity. The next point we are trying to address is we had talked about three conditions.



We said one is fluid statics, another is fluid hauled in a truck, hauled at a particular acceleration, and the third case is fluid is truly flowing through a conduit of some nature. So, this is the way it comes to the third condition, which is that fluid is simply flowing through a conduit. In that case we have to come up with an expression for acceleration that A right. A was initially 0 for fluid statics, A had a finite $a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ for fluid hauled in mass at an acceleration, but then the third case, that is when you have fluid itself, is accelerating. The example I have given a conical section where the velocity is increasing so which means fluid is accelerating.

So, in that for the third case, I said acceleration can have to be given by something called a substantial derivative upper case dv, upper case dt. So, what is that, why do I end up with that substantial derivative? If we try to see what it is? u is a function of x, y, z, and t. So, space and time. So, this is the velocity field let us say. So, we have a velocity field. Ideally, we should be calling it a velocity v.

So, let us call this V_A and this is again V_B. So, velocity is given by this will let us say to avoid confusion we write it in terms of capital V. Then what we see here is that if we talk about the velocity the u component of that velocity we see that the u component of the velocity is given by u = u (x, y, z, t) again. And here again, after some tim, $t = t + \Delta t$. You will see that this is a time t the fluid all the fluid particles were in this box differential volume where the space special coordinates were x y z. At some other $t + \Delta t$ the special coordinates are $x + \Delta x$ because over this time period the fluid particles have moved.

So, this is now $x + \Delta x$, $y + \Delta y$, and $z + \Delta z$. So, these are the new special coordinates over which the differential element is constructed. So, now, this is again here if we focus only on u component here also we will have, $u = u(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t)$. So, now, if we try to take a Taylor series expansion because dt is all these time the Δt is small. So, Δx, Δy, Δz all are small.

So, if we try to take a write

$$u(x + \Delta x, y + \Delta y + z + \Delta z, t + \Delta t) = u(x, y, z, t) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z + \frac{\partial u}{\partial t}\Delta t + \cdots$$

So, these are the first derivatives you have. I mean if you want to continue with these you will have del square u del x square delta x square by 2 factorial plus del square del square u del y square delta x delta y whole square by 2 factorial like that you will have all these additional terms would be coming. So, we are for the time being we are working only with this with the assumption that you have only the first order terms considered for Taylor series expansion.

So, this is what we end up with. Then what we see here is this: instead of this Δx, we can write Δx as $u\Delta t$ because u is the velocity in x direction. So, over time Δt the Δx the distance traveled in the x direction by these particles that would be equal to $u\Delta t$. So, if we write Δx as $u\Delta t$, Δy as $v\Delta t$, Δz as $w\Delta t$ and then this the other term remains. So, this is what we end up with.

$$u(x+\Delta x, y+\Delta y+z+\Delta z, t+\Delta t) = u(x,y,z,t) + \frac{\partial u}{\partial x}u\Delta t + \frac{\partial u}{\partial y}v\Delta t + \frac{\partial u}{\partial z}w\Delta t + \frac{\partial u}{\partial t}\Delta t + \cdots$$

$$\frac{u(x+\Delta x, y+\Delta y+z+\Delta z, t+\Delta t) - u(x,y,z,t)}{\Delta t} = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t}$$

So, this is if we set limit delta t tending to 0 this is essentially the acceleration as far as the x direction is concerned that is what the Eulerian acceleration we call it. So, this acceleration is not merely $\frac{\partial u}{\partial t}$ just what we see for particle acceleration. This acceleration has these additional terms and if you extend the same concept in y and z direction.

So, you will have a very similar expressions

$$\Rightarrow a_x = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t}$$

$$\Rightarrow a_y = \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w + \frac{\partial v}{\partial t}$$

$$\Rightarrow a_z = \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w + \frac{\partial w}{\partial t}$$

So, these are essentially a_x a_y and a_z these are the acceleration terms. So, why it is happening is that the velocity that we talk about, u v or w these, are not the velocity of a particle. These are the average velocity of all particles. As I said, that is the Eulerian concept that has to be accounted. And then, in some cases, this part of the expression is referred to as convective acceleration, and this part is referred to as local acceleration.

So, this a_x is written as $\frac{Du}{Dt}$, this is called the substantial derivative and that is how we differentiate the regular partial derivative of u with respect to t we differentiate we call it substantial derivative which is the sum of the convective acceleration and the local acceleration. And you have one thing you may like to note I mean why do I need a convective acceleration term what I understand is that a typical explanation provided by the authors who have developed this framework it is something like this. Suppose someone is standing on a bridge and it there is a there is a small revolute flowing below this and then there are fishes travelling. So, now, someone wants to know what is the velocity of the fishsitting on the top of the bridge. So, the position of the observer is fixed, observer is not travelling with the fish, observer's position is fixed on the bridge and from that position observer is trying to measure the velocity.

So, when it comes to measuring the velocity, the observer has to find two position vectors, one position vector at time t and another position vector at time t+Δt, and then find the difference in the distance traveled over this time Δt and then divide it by Δt to get the velocity of the fish. That the observer can do by sitting in its position fixed Eulerian framework is your position is fixed and then if you are asked to ask the observer to find out what is the acceleration of the fish. Then in that case observer has to find out what is the velocity of the fish at this location what is the velocity of the fish at this location subtract the two divided by the time taken to move from here to there and you get the acceleration. Now, the problem is when you need to find out the velocity of the fish at this location here by sitting in that position you cannot find out for that you have to move you have to find another reference frame to find out what is the velocity of the fish at this location and you have to find another reference frame to find out the velocity at this location. So, for that probably one way of doing it is probably some someone with a canoe has to sit there and measure the velocity of this.

So, some observer has to sit on the canoe and measure it another observer has to sit on the canoe and measure this velocity and then give that information to the person who is sitting on the bridge and then based on that he or she can calculate the acceleration. So, this one has to one has to one has to change the reference frame. So, I mean sitting when your position is fixed as is the case for Eulerian framework when your position is fixed you need to have additional terms to account for this aspect. So, that is why this convective acceleration is very important. In a moment, I will give you an example and where you will see that, for example, in a very simple case, I point out that conical section that I mentioned if I let us say this is the conical section.

So, here the fluid is accelerating but it is a non uniform flow, but it is a steady flow there is no time derivative involved the Q/A the velocity and here it is Q/A. So, velocity is continuously increasing because area is continuously decreasing as you move from inlet to outlet. So, this is definitely perfect example of non uniform flow, but steady flow. Unsteady flow means with time, the flow is changing here with time, flow is not changing. So, if with time flow is not changing then your $\frac{\partial u}{\partial t}$ will always be 0 fluid will not be accelerating.

So, if you are using Eulerian framework unless you have some additional terms. So, to say I mean referred as convective acceleration you will not be able to obtain the acceleration and if you still want to do it I mean you stick to it then I will do it by this then you have to travel with a canoe that is what we will go through an example and this becomes obvious.

For fluid particles in motion,

$$-\nabla p + \rho \vec{g} = \rho \vec{a}$$

Where, $\vec{a} = a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{k}$

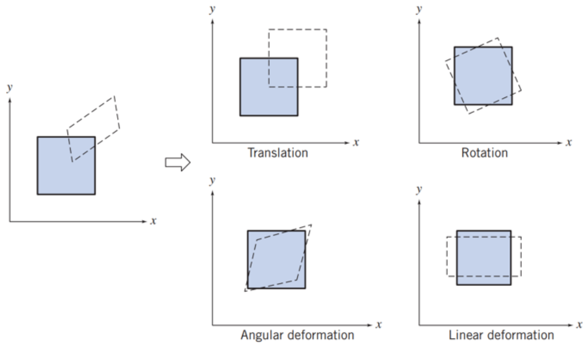
$$\Rightarrow \vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} = \frac{D\vec{V}}{Dt}$$

Where $\vec{V} = u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}$

You have here fluid particles in motion we already have talked about this acceleration and acceleration is a $\vec{a} = a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{k}$, but here in this case the acceleration becomes acceleration takes this form. So, that when we talk about the a_x, $a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$. So, like this similarly you have the a_y and a_z term in the compact notation this would be the acceleration and referred as the substantial derivative of velocity vector.

So, three cases we said one is fluid statics acceleration of 0 second case fluid is hauled n mass with an acceleration then you have a finite acceleration of simple form like this and when the fluid is accelerating in a conduit then you have to take into the substantial derivative. So, these are the three treatments we have. Now, with this understanding in place let us say we try to find out what all motions a fluid particle will undergo. One thing we just now said is acceleration. The acceleration is acceleration is a part of so, called translation that means, I have let us say at time t, I have the fluid element which is of shape like this and then at time t+dt we find that the fluid element has moved rotated deformed and all those processes have happened to it.

Motion of Fluid Particle



So, now the acceleration that I have talked about that is essentially translation that means, this is the initial one and then after $t + \Delta t$ it is simply it is everything shape and all these remaining same only the center has moved to some distance some Δx some Δy some Δz over time Δt. So, this translation is given by the expression that we had talked about. Other than this the one can have something called rotation that means, the fluid element itself around the center it is rotating you can see that this whole thing has rotated. So, now one thing is there: if to rotate a fluid element, you need a shear stress that will get into, but then this is one aspect of it fluid element gets rotated. The other aspect is fluid element get deformed, there is angular deformation.

In fact, that is what we discussed when we defined viscosity you must remember a fluid element over time Δt it got deformed. So, angular deformation and then rate of angular deformation equated with our what you have with the shear stress in Newton's law of viscosity. So, this angular deformation is also a part of this and the fourth case is linear deformation that means, the angle 90 degree remains same, but it was a square now it is a rectangle. The total area is conserved, but the shape changes.

So, but angle remains 90 degree. So, this is known as linear deformation. So, this is angular deformation this is linear deformation. So, what we will do next in the next lecture is we will go one by one through these translation, rotation, angular deformation, linear deformation and see if the entire thing happens how the movement of the fluid or motion of the fluid particle can be characterized. That is all for this module of the lecture.

Look forward to take this further in the next one. Thank you very much for your attention.