Momentum Transfer in Fluids

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Week-03
Lecture-12

welcome you to this lecture on Momentum Transfer in Fluids. We have been discussing about inviscid flow, and in particular,l in the last lecture, we discussed about continuity equation, which is a precondition for any fluid flow to take place. So, I will continue to continue discussing on that. What we mentioned at the end of last class was that $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0$. So, that is a condition any fluid flow has to satisfy.

Equation of continuity in differential form,

$$\boldsymbol{\nabla} \cdot \boldsymbol{\rho \vec{V}} + \frac{\boldsymbol{\partial \rho}}{\boldsymbol{\partial t}} = \mathbf{0} \qquad \left(\because \nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)$$

Case-I, Incompressible Fluid

$$\nabla \cdot \vec{V} = 0$$

Case-II, Steady Flow

$$\nabla \cdot \rho\vec{V} = 0$$

$$\Rightarrow \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

So, you can choose any velocity vector, but that will not be a velocity field for an Eulerian framework unless this condition is satisfied. So, now, this is written in a compact form you can see here this is the compact form where the grad is defined as $\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$. So, you can write it in compact form the continuity equation. Essentially in an equation of continuity in differential form, you could have had an integral form, I mean you could have had the area as Δx, Δy and you did not have to you need not have to write it in this way I mean if you want to write the continuity equation in an algebraic form, but this is how if you give a velocity field one has to satisfy this continuity in differential form that is a must.

Now, when the fluid is incompressible, fluid is incompressible means density does not change, change with time, change with space nothing density of water constant 1000 kg per meter cube. Maybe there is many skew change with pressure. There is some amount of compressibility possible, but for the time being we are not considering that we are ignoring it. So, density is constant that means, that this there is no scope of density varying with time. So, in that special case of incompressible fluid. So, density is constant, and not only this is 0, density will come out of this.

So, instead of density here density will come out. So, it would be

$$\rho\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]=0$$

So, then, in other words, ρ is some constant. So, it is you are essentially saying $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$. So, this is the case for incompressible fluid flow that is 0. So, this is what is mentioned. Also, there is a special case of steady flow when the density does not change with time, but density may change with space. That could be a special case where $\frac{\partial \rho}{\partial t}$ became 0, but the left hand side, this expression will remain as it is. So, that is what you see here that is for the steady flow. So, if you are working with incompressible flow one has to make sure that $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$. What are u, v, and w? It is $\vec{V}=u\hat{i}+v\hat{j}+w\hat{k}$.

These are the velocity in x, y, and z components, but mind it, this is the average velocity of all particles inside that differential volume, and these velocity field is treated as continuously changing in space and time with no discrete change. So, with this understanding this continuity equation in fact, I can give you some equation, and I can ask you that whether that is whether flow is feasible or not. So, if someone comes up with an expression saying whether this velocity field is acceptable as a fluid flow, whether you will you will accept that or not. So, the continuity equation defines that.

- For a two dimensional flow in the *x-y* plane, the *x* component of the velocity is given by *u=Ax*. Determine a possible y component for incompressible flow. How many *y* components are possible?

$$\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$$

For incompressible flow in rectangular coordinates, the equation of continuity simplifies to

$$\nabla \cdot \vec{V} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

For two dimensional flow, $\vec{V} = \vec{V}(x, y)$

$$Hence,\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow -\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -A$$

$$\Rightarrow v = \int -A\, dy + f(x,t)$$

$$\Rightarrow v = -A\, y$$

$$\vec{V} = u\,\hat{\imath} + v\,\hat{\jmath}$$

$$\Rightarrow \vec{V} = Ax\,\hat{\imath} - Ay\,\hat{\jmath}$$

In fact, one example problem is given here for a two-dimensional flow in the x-y plane. This velocity is given by u=Ax. So, the x component of the velocity is Ax and determines a possible y component for incompressible flow. So, incompressible flow that means, we have what equation I mean in the compact form we can write it, but essentially it is boiling down to this $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. There is no w component it is a two dimensional flow. So, $\frac{\partial w}{\partial z}$ is not appearing.

So, this is the continuity equation that one has to satisfy, and on top of that is given that u=Ax. And what you are asked is determine a possible y component for incompressible flow. So,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

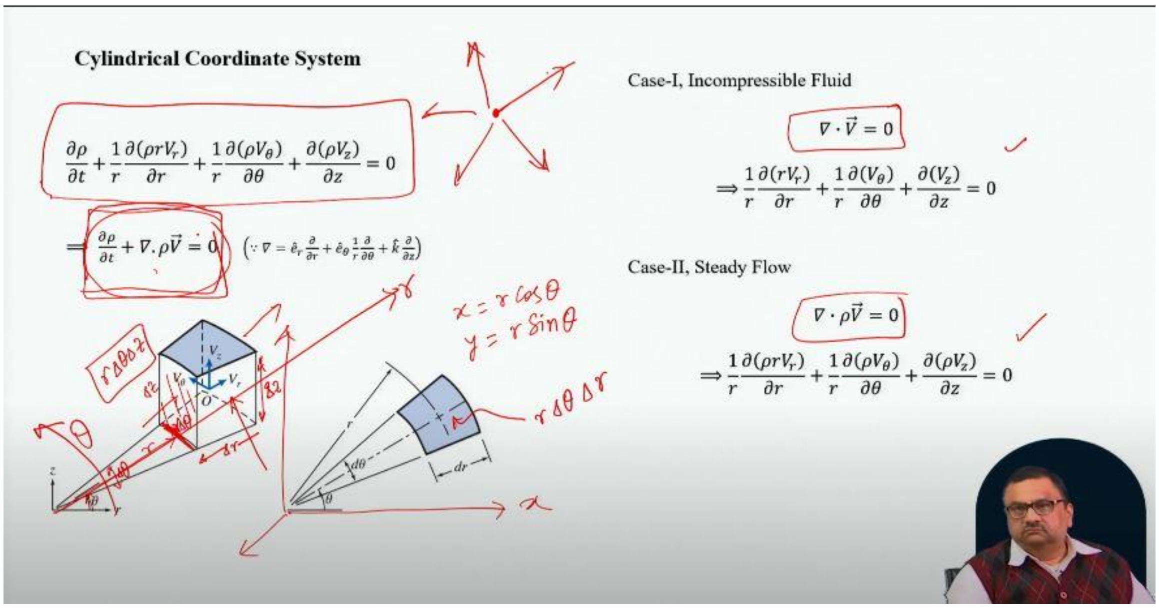
$$\Rightarrow -\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -A$$

$$\Rightarrow v = \int -A\, dy + f(x,t)$$

$$\Rightarrow v = -A\, y$$

So, the velocity would be, if this is 0 if this $f(x,t)$ is equal to 0 then v would be equal to $-A\,y$. integration of this will give you $-A\,y$. So, in that case the velocity field $Ax\,\hat{\imath} - Ay\,\hat{\jmath}$ will satisfy the conservation equation for sure. So, this is one way, or I could have given you this expression, and I could have asked you whether this satisfies the continuity equation or not. So, I can see here if we go by this equation then $\frac{\partial u}{\partial y} = A$, derivative of Ax with respect to x is A, $\frac{\partial v}{\partial y} = -A$. So, $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0$.

So, that becomes 0. So, $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0$. So, the continuity equation is satisfied. So, this type of checking one needs to do instead of working with any arbitrary function for the velocity field. Now, the continuity equation that we mentioned we need not have to that is essentially for a rectangular system, but in many cases, in fluid flow, you will see that the rectangular or Cartesian coordinate system may not be always helpful because you have symmetry in a certain way in the problem and you want to leverage that and in those cases cylindrical coordinate system works, the cylindrical coordinate system turns out to be very helpful.



So, in that case what you do is so, first of all what is the cylindrical coordinate system? We had talked about x, y, and z coordinates. So, we have this as the x-axis; let us say this is my x-axis, and this is my y-axis. Actually, this is referred to here as z. So, this is the z-axis. So, y-axis would be in other direction maybe this direction. So, what we see here is I pick a differential element.

So, earlier, I picked up a differential volume of dimension Δx, Δy, and Δz, where the center of the differential volume is located at coordinate x y z. Instead of that I pick up a differential volume which has a dimension. In fact, I was right z was perpendicular to the screen. You can see z is perpendicular to the screen. So, this the way this is working this should be y.

Let us first look into this. This is the differential element in cylindrical coordinate system. So, what we see here is that this distance to the center here is, let us say, r. This is r and this angle it makes is known as θ. So, in this case, we will write $x = r\cos\theta$, and $y = r\sin\theta$.

So, these are the x and y and the z direction will remain z as before. So, what you do in this case is if the differential element takes this type of shape. So, then, as far as this surface is concerned, what would be the area? This side would be if this is θ and this angle is dθ this or Δθ. So, this would be rΔθ, arc length would be rΔθ.

This side is Δz, this side is the z axis. So, this area would be equal to rΔθΔz. So, that is the area we are talking about. Last time we had what? we had Δy Δz Δx Δz. This time we have rΔθΔz.

Similarly, what would be this area? This blue shaded area. This area would be dr in this dimension, and this dimension it is rdθ. So, this area would be equal to rΔθΔr. You can write it rdθdr I mean that is how it is. So, now, if you do the same, if you apply the mass conservation just the way we have done flow in through this area, flow out through this area, flow in through this area, how much would be this area then? This area will be straightforward because I see this is Δr and this is Δz.

So, this area is ΔrΔz. So, flow in through this, flow out through this, flow in through this, flow out through this. So, if you do that same conservation equation that we had done in the Cartesian system now, instead of that, if you do in the cylindrical system, we end up with the continuity equation, which takes the form of this. So, we have this takes the form as this. So, again in compact notation, we will write this only. This is the compact notation we used earlier. Also, in the Cartesian system only thing is the definition of grad. In the Cartesian system, it was $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$.

Here in this case the definition of grad is

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}$$

$\hat{e}_r$ that is the unit vector in r direction $\frac{\partial}{\partial r}$ here it is $\hat{e}_\theta$, θ has a direction mind it r is this direction, this is r, θ has a direction, this is theta θ. So, θ also has a unit vector $\hat{e}_\theta$. So, $\hat{e}_\theta$ is the unit vector in θ direction and then multiplied by the $\frac{1}{r}\frac{\partial}{\partial \theta}$ that is what you have and $\hat{k}\frac{\partial}{\partial z}$ it remains as it is. So, the way we are putting it as if the definition of grad is changing, but the basic equation remains same, as we pointed out.

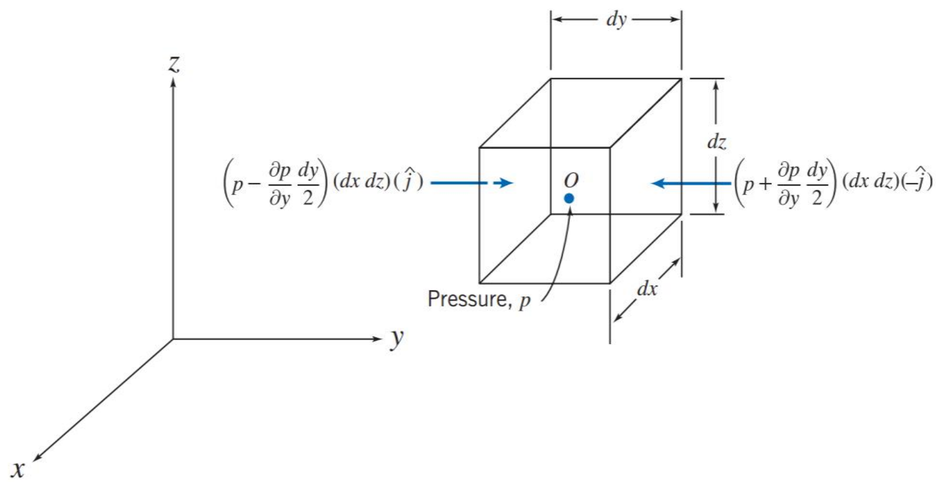
So, that is what we must remember. Now, what is the advantage of going to the cylindrical coordinate system? For example, if we have a fluid motion which is in circle or a fluid motion which is moving radially outward. Say, let us say I have a fluid motion, which is sort of a source that means the fluid is emanating from a point source. So, when we have this, we have certain symmetries or fluids moving in circles. So, we can have certain symmetries arising which we can leverage and here we have all x component y component z components become a little cumbersome with algebra, but that can be much simplified if we choose to use cylindrical coordinate or we can cancel certain terms and work on a more simplified approach. So, again, once again, for incompressible fluid, the same similarly, we have an equation, and in the case of steady flow, we have.

So, these are all applicable for cylindrical systems. So, cylindrical coordinate system, just like the Cartesian system, we have an equation system. Here, the basic equation framework remains the same as I mentioned, but the definition of grad changes, and accordingly, the detail format changes. What you do next is so, with this understanding of continuity, once we understand that a continuity will be working and one has to satisfy, then we would like to see what all forces are operational on the fluid. So, same differential element we have picked up x y z and then we have a differential element of size dx dy dz and we try to find out what all forces are acting on these fluid. Of course, our treatment is, as I mentioned before that it is going to that means, we will be not discussing about the shear stress, we will be discussing about pressure, and we will be talking about pressure forces.

Now, first of all, there is a certain body force possible on this differential volume. Of course, the most obvious body force that we work with is gravity. So, body force here is given as acceleration due to gravity multiplied by the mass, and here mass is mass of this differential volume. So, dm and this dm mass of differential volume is ρ dV, dV has a unit of kg/m^3 that is the unit of volume and dV has a unit of m^3 that is the unit of volume of the differential element and ρ is the unit of density whose unit is kg/m^3. So, you have ρ unit is kg/m^3, dV has unit of m^3. So, when you take the product this gives me kg.

So, this ρdV is giving me kg which is dm and then that multiplied by acceleration due to gravity gives me the body force. Now, this dV happened to be dx dy dz the volume of this differential element is dx dy dz. So, this is the body force acting on a differential element that I have mentioned here. How about the surface forces? If we do not have the shear stress, I would have on this differential element there would be surface forces in the form of pressure acting on that differential element. Last time you may recall that δxx δyy δzz, they were acting outward through the faces.

But now, since the pressure is acting on a point, so pressure would be acting towards the center here. So, now, I have pressure at point O, point O the pressure is given as P, O is the center of this differential element. So, we need to find out what is the pressure at the left face and what is the pressure at the right face because they have to balance. Similarly, because pressure in the x direction has to be balanced by another or the force in the x direction has to be balanced by another force in the x direction. Similarly, force in the y direction has to be balanced by the force in y direction.



So, if we look at what would be the force acting as far as this left face is concerned. First of all, pressure is P here, and so the pressure at the left face would be again we do the same thing as we are done in the case of mass conservation, in the case of ρ and u. We said that at the center, the density is ρ, and at the left face, it is ρ at x, ρ at x, y, z, and at the left face, it is ρ at x –dx/2 because this distance is dx/2. So, this is, once again, I am having a Taylor series expansion and ignoring higher order terms with the assumption that pressure changes from the left face to the right face linearly since the dx is small. This is a differential element because later on, we will write Δx tending to 0.

So, it is not less than p and more than p it depends on what sign you have for $\frac{\partial p}{\partial y}$. So, on the left face it is p - $\frac{\partial p}{\partial y}\frac{dy}{2}$. Essentially p if you write at the left face that p at x, y, z what you have minus $\frac{\partial p}{\partial y}\frac{dy}{2}$ then you have other higher order terms and here similarly you have p+ $\frac{\partial p}{\partial y}\frac{dy}{2}$ plus other higher order terms. So, what you do is here you have this is the pressure applied on the left face this is the pressure applied on the right face assuming pressure varying linearly from left to right. What is the area over which this pressure is acting? This is dx dz because this side it is dx and this side it is dz.

So, this area is dx dz. So, that I have multiplied here dx dz, and here also I have multiplied dx dz. So, what does this give me? Pressure is in unit of N/m^2. dx is unit of meter dz is unit of meter.

So, this is meter into meter. So, the product is Newton. So, this is the force that is acting as far as far as the left face is concerned, and these forces act in the positive y direction. So, that is why I have $\hat{j}$ that is the positive y direction. Here in this case this force is acting in the negative y direction because y direction is here. So, it is acting in the is acting in the negative y direction.

So, that is why I have this is as $-\hat{j}$. So, what I need to do if I really want to find out the force balance, if I do not bother about the shear stress, and at the same time I want to do the force balance. In that case, what I will do is I have to find out this is the net force acting in positive z direction, this is the net force acting in negative z direction. Same thing applies to this face, same thing applies to the this face and from that face outward, same thing applies to the bottom face and the top face. So, we have these surface forces 6 different 1, 2, 3, 4, 5, 6. So, on 6 faces I have these surface forces acting I will call them F_s.

So, I will have the summation all the F_s. On top of that, I have the body force, body force just now, talked about, which is the dx dy dz, which is the volume of this differential element multiplied by ρ. So, unit of ρ is kg per meter cube, volume of this element is meter cube. So, that is kg ρ into dx dy dz unit is kg that is the dm mass multiplied by acceleration due to gravity. So, that gives me the gravitational or the body force. So, I have these 6 surface forces and I have one body force.

So, that is the net force acting if I do not have any shear stress acting on this. Either fluid is moving in mass, or fluid is static, or I mean one layer is not sliding against the other. So, these forces are acting on this fluid element, and then this has to be equal to as per Newton's second law of motion, this has to be equal to mass into acceleration. So, this fluid will accelerate because of these surface forces that I mentioned here and the body force that I mentioned there. So, when I equate surface force plus body force, that is the net force acting on this differential element, and because of this net force, this fluid will undergo acceleration.

So, by Newton's second law force will be equal to mass into acceleration. Now, this acceleration of the fluid if we are talking about a fluid statics problem in fact, that is something which we will be doing down the line in this course. In the case of the fluid statics problem, we will show that acceleration is 0, and there could be another case where the fluid is placed in a tank, and the tank is hauled, let us say. So, in that case I will have an acceleration for the fluid, but that acceleration is slightly different not slightly in a major way it is different from the fluid that is accelerating as it flows through some conduit or through some place fluid itself is accelerating that is one thing, because of pressure gradient because of other reasons. So, that is one thing, or I can put the entire fluid in a tank and put it, and the truck is accelerating. That is also acceleration. That is another case, the second case, and the third case is fluid is simply static.

So, acceleration is 0. So, we will go by these 1 2 3 these 3 cases and we will show we will apply Newton's second law at least fluid statics acceleration is 0. So, force is total force is 0 and from there we will make some conclusions derive some equations, but the other 2 cases we will talk about and we will discuss and as part of these inviscid flow. So, what we are trying to do at this time is trying to find out the net force acting on this fluid element when shear stress is not present I am not considering viscosity effect, but we are trying to find out under this condition what would be the pressure profile what would be the fluid flow. So, that so, these details we will figure out. So, that is what I will continue I will draw this force balance and apply Newton's second law and see what we can gather out of this exercise that is all as far as this lecture module is concerned I will continue this discussion in the next lecture. Thank you very much for your attention.