Momentum Transfer in Fluids

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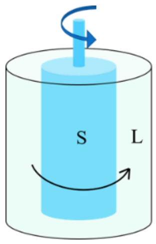
Lecture-10

Good morning. We are going to start with continuation of shell momentum balance, but I will show you some results, some situations in which you would feel that imagining a shell for such a complex geometry could be a problem. Secondly, so far we are dealing with velocity being a function of only one variable. It could be one of the special coordinates x or y or z or it could be in some situations, for transient cases the velocity could be a function of time. But what if the velocity is a function of two variables, two independent variables. It could be x and y, two-dimensional flow or it could be x and time. So, under those special circumstances it is extremely difficult to use the shell, imagine a shell and make a balance of momentum.

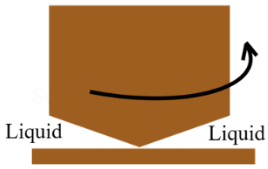
So, let us look at some of the examples. Here what we have in this case is one cylinder which is stationary and the other cylinder which is being rotated. So, you could see that there is going to be, it is a cylindrical coordinate system. The velocity here is going to be a function of r, where r denotes the radial distance. This is the r direction. So, velocity is going to be v_θ the only nonzero component, it is going to be a function of r. It may not be a function of z or of theta.

Situations of Increasingly Complex Geometry – Shell MM Balance??



One cylinder rotates, the other stationary

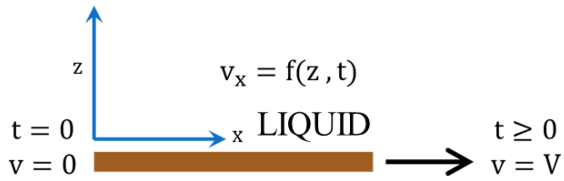
Cone and plate viscometer



So, this is one situation where it is slightly difficult to imagine what could be a shell for such case. And this next figure that you see is a cone and plate viscometer which is quite common specially for the measurement of high viscosity liquids. Now, here we have a cone and the apex of the cone almost touches the plate, the flat plate and the liquid is kept in the intervening space. So, what we have is, we have liquid in here and the top cone, the cone is being rotated with a fixed angular speed. Now, the torque required to move this cone with a certain velocity is measured and from there the viscosity is going to be correlated with the torque, with the measured torque.

So, this is how the unknown viscosity of viscous liquids are measured. Now, again what is going to be the shell in such a case, here you could see that the velocity is going to be a function of, it's distance, it's distance from the straight plate, it's distance from the surface of the cone and it could also be a function of the distance in the r direction. So, there could be multidimensional effects as well. So, the shell momentum balance as we have practiced so far will may not work in this situation. Then there is another class of, I would say problems where let us say this is a flat plate and I have a liquid on top of it.

A stationary plate
suddenly set in motion



So, initially the liquid and the plate is at rest and then suddenly at time t equal, greater than equal to 0, at time t equal to 0 the bottom plate starts to move with a constant velocity of capital V. Now, as the solid plate starts to move, it is going to impart a velocity in the adjoining liquid close to the solid surface. At the next instant, this velocity of the liquid initiated by the motion of the plate will be transmitted in the plus z direction. So, initially only maybe this layer is moving with certain velocity. After sometime the viscosity will drag the liquid layer above it and so on. So, a motion of the bottom plate which is in contact with the fluid will make the velocity of the liquid in the x direction as a function of not only the distance from the plate, but also as a function of time.

So, it is a perfect example of 2-dimensional flow where a stationary plate is set in motion and that is going to initiate a motion in the fluid above it. So, the shell momentum balance will definitely not work here. So, in order to solve such problems, we would like to have a generalized approach and this generalized approach, before we get into the generalized approach, we have to get the concepts of certain important parameters, certain important definitions. And then these definitions of partial, total and substantial time derivative derivatives, they are used to derive the equation of continuity which is nothing but a statement of the conservation of mass. And more importantly the equation of motion, which is Newton's second law for an open system.

And once I have this equation of motion, certain restrictions or simplifications can be suggested which would lead to Navier Stokes equation. And this Navier Stokes equation can be used for any situation and you can write the Navier Stokes equation for the right component in the direction of the velocity, in the direction of the motion and then try to solve it analytically or numerically in order to get an idea of the entire flow field existing in the fluid. So, Navier Stokes equation is extremely useful and a fundamental equation that allows us to calculate, to measure the flow field, to calculate the shear stress and it creates an important aid to the design of many equipment, many formations which are in contact with a liquid or a fluid. And if I impose the condition that the fluid that we are dealing with is not viscous, it is inviscid fluid with 0 viscosity, then from the Navier Stokes equation, the Euler's equation would follow. And one of the examples of the use of Euler's equation as you know is Bernoulli's equation.

So, starting with the concept, the equation of continuity, motion, Navier Stokes equation and finally Euler equation, we are going to use all these in our subsequent classes to get a fair idea about how the velocity, the shear stress field changes in a moving fluid for any complex geometry. So, that is going to be our task for the next few classes, but before that we need to clarify certain concepts for example, the before we reach to the equation of change. Now, in order to give you an idea of the different derivatives which are commonly used in fluid mechanics, let us assume that you are standing at a very busy intersection in a major city where there are 4 or 5 roads converging to that point and they are going in different directions. So, I ask you that, you stand right at the centre point, at the central island, where you can see all the cars that are approaching you or passing by you. And I request you to count the number of white cars which you can see in all, among all those cars which are passing you by.

Now, you are standing at a fixed point. So, the origin of the coordinate system is fixed and you are counting the number of the white cars. The time variation of the number of white cars that you measure, that is known as the partial time derivative. In this partial derivative the origin of the coordinate system that x y z, your location is constant. You are simply counting the numbers of white cars passing you by. Now, let us assume that after obviously, after standing there for quite some time you get bored.

Partial time derivative: $\frac{\partial C}{\partial t}$ (x,y,z constant)

So, you start to move around you decide to just go to some other place and but since I have requested you to count, keep on counting the numbers while moving from one point to another, you are still counting the number of white cars passing you by. Now, you have a velocity of your own your, velocity has 3 components dx/dt, dy/dt and dz/dt. So, with your velocity superimposed on the motion around you, the number of cars, number of white cars that you calculate would be known as the total time derivative. So, in here this dx/dt, dy/dt and dz/dt are the velocity of u as you decide to move in any random direction. So, obviously, the numbers dc/dt would be different from del c/del t.

Total time derivative: $\frac{dC}{dt}$

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial C}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial C}{\partial z}\frac{\partial z}{\partial t}$$

So, the total time derivative, the number of white cars when you are in motion that you have counted would be different from the number of cars when you are stationary. Now, there is a third kind, let us say after walking for some time while still counting the numbers you have decided that I have had enough. So, let us flow with the cars, let us move with the average velocity of the cars around me at any given point of time. Mind it is the average velocity. And since it is average velocity, there will still be, still some cars which will pass you by or some cars which you will pass. Now, while moving with the average velocity of the fluid of the cars around you, are still counting the numbers, but now your velocity components are $v_{x,}v_y$ and v_z. Those are the fluid velocity, the velocity of the cars.

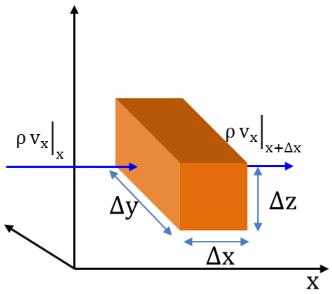
Substatial time derivative: $\frac{DC}{Dt}$

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + v_x\frac{\partial C}{\partial x} + v_y\frac{\partial C}{\partial y} + v_z\frac{\partial C}{\partial z}$$

So, to say around you and the number of white cars that you calculate at that condition is known as substantial time derivative or more importantly it gives you a better picture when we say that it is a derivative following the motion. So, the substantial derivative is therefore, the derivative following the motion where the velocity of u will simply be equal to the velocity of the fluid around you. So, these three derivatives are going to be important in our understanding of equation of motion, equation of continuity and so on. Conceptually they are the same, but the way they are expressed is different. A fixed axis, an axis moving with a constant velocity and at a coordinate system which is moving with the velocity of the fluid around it. So, with these three concepts clearly understood, now let us try to move and find out what is going to be the equation of continuity or equation of conservation of mass.

So, we are going to derive the equation of continuity. I am not going to show you all the steps, this is the derivation, is there in Bird Stewart Lightfoot. You can take a look at it, but it is the application of these equations and understanding of the different terms of the equation which are going to be important specially when we deal with the equation of motion. So, in order to derive the equation of continuity first we state the obvious, the conservation of mass and what does conservation of mass tell us that the rate of accumulation of mass inside a control volume must be equal to the rate of mass in minus rate of mass out. So, we need to first define a control volume fixed in space, identify the phases through which mass can come in or leave the control volume. As a result of this inflow, net inflow, that means, inflow minus outflow there is going to be some accumulation or depletion of mass inside the control volume. So, if we can express that in terms of quantities that we know, in terms of the velocity, in terms of the density, in terms of the area then we have our conservation equation written in a compact form, written in the form of a differential equation which would give us the equation of continuity.

Rate of mass ACCUMULATION = Rate of mass IN – Rate of mass OUT



In order to do that, let us assume that we have a cubical box whose size are del x del y and del z. Now, when we talk about the x face, the x face is the area which is perpendicular to the x direction. So, this is the x face and as you can see the x face has area equal to delta y times delta z. Similarly, the area that you see over here is the y face which has area of delta x and delta z and mass is going to come in through the x face and leave the face at, located at x plus delta x. Similarly, the velocity in the y direction will bring some liquid in here and it is going to leave out of this and same is true for z face.

So, the liquid, the fluid will enter through these faces at x, leaving at x plus del x, y and at y plus del y and so on. So, let us first identify what is going to be the terms that would indicate the total flow of liquid through each of these faces. So, rate of mass in through the x face and x face is this one. So, the area of the x face is del y del z multiplied by the velocity at x would give me the volumetric flow rate. So, meter square, meter per second and kg per meter cube.

Rate of mass IN through x-face: $\rho v_x|_x \Delta y \Delta z$

So, the product of these three essentially gives me the kg per second that means, whatever be the mass that comes in through this face. The rate of mass out through the x face would simply be everything remaining same except the velocity is evaluated at x plus delta x. So, that is going to be the only difference in this situation. So, imagining a control volume, putting it right in the flow field and identifying the terms with which the mass can come into the system or leave the system is a very useful tool in order to obtain what is the total amount of mass being added to the control volume. So, this is a control volume which is fixed at space, that means, the origin over here is stationary, it is not moving at all with time.

$$\text{Rate of mass OUT through x-face: } \rho v_x|_{x+\Delta x}\Delta y\Delta z$$

Similarly for other 2 faces: y and z

The same approach can also be used to obtain what is the momentum that comes in to the control volume. Now, if you look at this one. So, this one is the rate of mass in through the x face. Now, rate of mass in through the x face can be multiplied with velocity in order to find out what is the momentum that comes in through the x face. So, once we have this control volume, the different faces, the amount of mass being added, we if figure it out, then converting it to momentum is going to be straightforward.

But once again the mass that we are talking about, the momentum that I have explained subsequently are all due are convective process. The mass comes in because of the velocity associated with the fluid which enters through the x face. Similarly, when we multiply that with velocity again in order to obtain the momentum, that is convective transport of momentum through the x face. But we understand that there could be, apart from convective momentum there could be diffusive or molecular transport of momentum as well that we have to think about. But let us discuss that when we talk about the equation of motion.

Right now, the rate of mass in and the rate of mass out through the x and the x plus del x face, this should be x plus delta x. So, x plus del x face, x plus del x face is clear. So, we can write the similar terms for the other two faces, that means, the y face and the z face. So, for the y face if I just give you the one this simply would be ρ times v_y evaluated at y then multiplied by the area of the y face, which is delta x delta z. So, this is going to be the in term and the out term would remain the same except this y is going to be replaced by y plus delta y.

Similarly, for the z face you can write the $\rho\ v_z$ at z multiplied by del x del y where del x del y is the area of the z face and the area that goes, the mass that goes out through z plus delta z face would simply be where the velocity at that location is considered. So, as a result of all these inflow and outflow, there is going to be some accumulation of mass inside the control volume. So, what is the control volume? It is del x del y del z and the mass is changing with time which means that the density is changing with time since my size of the control volume is fixed. So, del x del y del z times del ρ del t would give us the time rate of change of mass, quantity of mass inside the control volume. So, I have the in, all the in terms, the out terms and the accumulation term.

$$\text{Rate of mass ACCUMULATION: } \Delta y\Delta z\Delta x\frac{\partial\rho}{\partial t}$$

$$\Delta y\Delta z\Delta x\frac{\partial\rho}{\partial t} = \Delta y\Delta z[(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}] + \Delta x\Delta z\left[(\rho v_y)\big|_y - (\rho v_y)\big|_{y+\Delta y}\right] +$$

$$\Delta x\Delta y[(\rho v_z)|_z - (\rho v_z)|_{z+\Delta z}]$$

So, now, I can write the equation that where the left-hand side is the accumulation term that I have defined. Del y del z this is the amount of mass coming in through the x face, amount of mass going out through x plus delta x face for y, y plus del y, z and z plus delta z. So, this is a difference equation. We know what we have to do is divide both sides by del x del y del z and take in the limit when del x del y and del z all approach 0. So, if that happens then we are going to get the differential form of the equation as del ρ del t, the time rate of change of density would be equal to del del x with a minus sign $\rho\ v_x$, $\rho\ v_y$ and $\rho\ v_z$.

If, $\Delta y\ \Delta z\ \Delta x\ \rightarrow 0$

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right]$$

Now, if you think more about what is ρ times v, ρ is kg per meter cube, v is meter per second. So, the so, this is going to be kg per meter square second. So, this is the mass flux. Now, this equation, the one that we have written over here, this equation can be made, can be expressed in a more compact form where del ρ del t is simply going to be minus of del of ρ v. So, this is the compact vector form of the conservation of mass equation which is also the equation of continuity.

$$\frac{\partial \rho}{\partial t} = -(\nabla . \rho v)$$

So, this is the mass flux as I have mentioned before. Now, this equation can also be expanded a bit because where I just find out, I just figure out what is del del x of $\rho\ v_x$. The first term is going to be v_x del ρ del x, the second term is going to be ρ del v del x. So, it this is simply follows from the differentiation. Now, I could take all the ρ all the I could take every term containing v_x on the left-hand side.

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right]$$

$$= -\left[v_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial v_x}{\partial x} + v_y \frac{\partial \rho}{\partial y} + \rho \frac{\partial v_y}{\partial y} + v_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial v_z}{\partial z}\right]$$

So, these terms are taken to the left-hand side to have del ρ del t plus v_x del ρ del x and so on would be equal to the right-hand side. Now, if you have followed the definition of the derivatives that we have discussed this is nothing, but the substantial derivative of density. I will go back quickly to the definitions that we have presented at the second slide and there you could see that what is the substantial time derivative. So, the substantial time derivative is $v_x\ v_y\ v_z$ multiplied by del c del x del c del y and del c del z. Now, if I go back to my equation that we have derived in this specific case here also I have, I see the same thing.

$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = -\rho\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]$$

$$\frac{D\rho}{Dt} = -\rho(\nabla v)$$

So, the right-hand side, the entire right-hand side is nothing, but the substantial derivative of density. Substantial derivative of density so, it is del d dt of ρ is equal to minus ρ times del v. So, this is another form of equation of continuity which can be used depending on whatever is convenient for a specific problem. So, the continuity equation essentially as I said conservation of mass, it can be expressed in times of del ρ del t or in terms of d ρ of dt and the, if we further assume that it is an incompressible fluid. An incompressible fluid is characterized by having a density having its density constant.

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} &= -(\nabla . \rho v) \\ \frac{D\rho}{Dt} &= -\rho(\nabla v) \end{aligned}\right\} \text{Continuity Equation}$$

So, if ρ is constant then this specific, the two equations that you see on the slide can be modified further which would simply tell us del v equals 0 or the bottom part is simply going to be this. So, essentially if del of v is 0 and you plug it in here. So, it is d this part is going to be 0. So, this is the differential form of continuity equation, conservation of mass for a fluid that is incompressible. This is the differential form, now if it is a differential form there has to be an integral form which we will discuss later on, but just for complete witness, I am going to write this, is going to be 0 which is where A is the area, v is the average velocity of the fluid passing through that area.

$$\left.\begin{aligned} (\nabla v) &= 0 \\ \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] &= 0 \end{aligned}\right\} \begin{array}{l}\text{Incompressible Fluid} \\ \rho = \text{Constant}\end{array}$$

So, if you expand this, it is going to be A 1 v 1 plus A 2 v 2 plus A 3 v 3 and so on would be equal to 0 at would be 0 for an incompressible fluid. However, there has to be some word of caution in here is what am I going to write, what sign I am going to write for A 1 v 1. The convention or the, what is done is that if the flow is into the control volume it is going to be negative and if the flow is out of the control volume it is going to be positive. So, while using A 1 v 1 A 2 v 2 this formula you have to be careful that if the flow is in, A 1 v 1 is going to be negative, if the flow is going to be out of the control volume this term is going to be positive. Now, why is that because if you understand that for any area if I talk about this area the area vector is always pointed out of this.

So, this is your A i. Now, if you have the velocity in here, the velocity is into the control volume through the surface area, then this A i v i will be negative. Whereas, if the v i is out of the control volume, then A i v i is going to be positive. So, this is the essence of continuity equation both for, both for this, for an incompressible fluid, for a fluid which may not be incompressible equation of continuity in the differential form, equation of continuity in the integral form. So, this is the integral form of continuity equation and we use this equation many times specially when dealing with flow coming in with multiple inlet and outlet in a control volume. So, for that case the formula to be used is A 1 v 1 plus A 2 v 2 equal to 0.

One point to note here is that this v 1 is not the point velocity. Since, we are dealing with the control, dealing with the integral approach, this v 1 is essentially averaged over A 1 the entire area A 1. So, this also highlights the difference between the integral approach and the differential approach. So, far we are doing differential approach, but when we go for integral approach, there is going to be some sort of an approximation and here you can clearly see where does that approximation come from. The velocity here in the differential approach I mean the equation that I have written for the differential approach is valid for every point.

Whereas, the expression that I have used for the integral approach over here, the velocities are assumed to be constant over the entire A 1. The closest you can get to that approximation is when the velocity is the average velocity. So, for average velocity, using average velocity, the expression that A 1 v 1 summation A 1 v 1 summation A i v i to be equal to 0 for a fluid of constant density incompressible fluid that is going to be valid. So, with this I stop here and in the next class I am going to talk more about the equation of motion and how Navier Stokes equation can be derived from, can be obtained from the equation of motion and subsequently Euler's equation. Once we have our idea of equation of continuity and the Navier Stokes equation then I am going to show you how easy it is to use Navier Stokes equation for solving problems.

So, I will start with the problems that I have already solved using a shell momentum balance and then slowly go towards slightly more complicated problems where you can truly appreciate the beauty of Navier Stokes equation. So, the next class is going to be on the conceptual, not stepwise conceptual derivation of Navier Stokes equation. Thank you.