Chemical Engineering Fluid Dynamics and Heat Transfer

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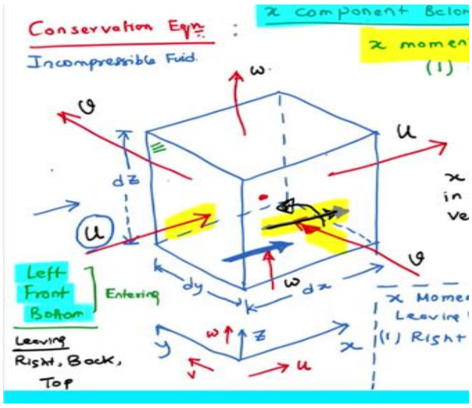
Lecture - 09
Conservation Equation 04 - Conservation of Momentum -02

So, welcome back. we will continue with momentum balance.



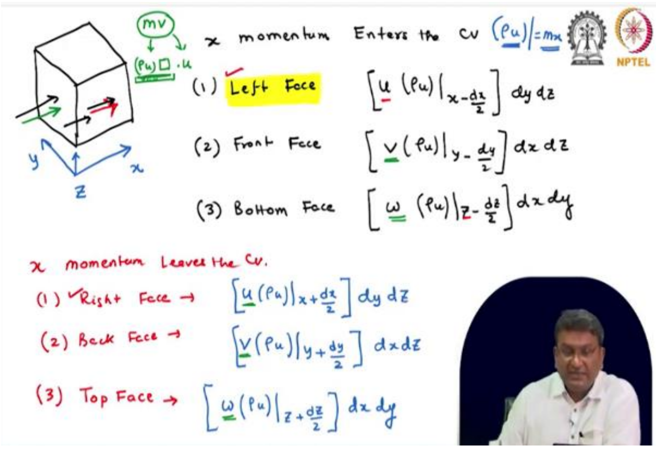
In the last lecture we have discussed about the x momentum which is the mass flow due to the x component velocity and it is entering from the left face due to the x component velocity and from the front face due to the y component velocity and from the bottom face to the z component velocity to the control volume.

x momentum entering the control volume:

Due to the x component velocity, through the left face = $\left[u.(\rho u)|_{x-\frac{dx}{2}}\right].dy.dz$

Due to the y component velocity through the front face = $\left[v.(\rho u)|_{y-\frac{dy}{2}}\right].dx.dz$

Due to the z component velocity through the bottom face =$\left[w.(\rho u)|_{z-\frac{dz}{2}}\right].dx.dy$



And similarly, the x momentum leaves the control volume through the right Face due to the x component velocity, the Back Face due to the y component velocity and Top Face due to the z component velocity.

x momentum leaves the control volume through the

Right face: due to the x component velocity = $\left[u.(\rho u)|_{x+\frac{dx}{2}}\right].dy.dz$

Back face: due to the y component velocity = $\left[v.(\rho u)|_{y+\frac{dy}{2}}\right].dx.dz$

Top Face: due to the z component velocity = $\left[w.(\rho u)|_{z+\frac{dz}{2}}\right].dx.dy$

Volume of the control volume = dx dy dz =V

The net x momentum across the control volume

$$\frac{\partial(\rho u^2)}{\partial x}dxdydz+\frac{\partial(\rho uv)}{\partial y}dxdydz+\frac{\partial(\rho uw)}{\partial z}dxdyd=\left[\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}\right]V$$

$$=\rho V\left[u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\right]$$

Since the flow is incompressible, the term $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$,then the above eqn becomes,

$$=\rho V\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$$

This terms you can identify as the convective acceleration terms in the substantial derivative.

Next, we are looking into the y component momentum balance:

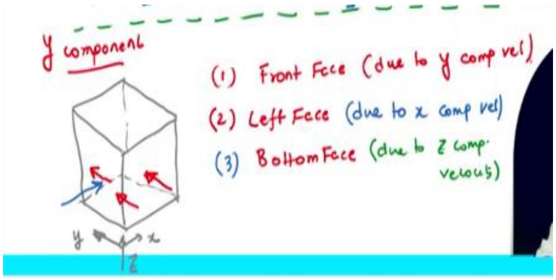
Similar to x momentum, y momentum is entering from the front face due to the y component velocity and from the left face due to the x component velocity and from the bottom face due to the z component velocity to the control volume.

y component momentum entering the control volume through

1.Front face due the y component velocity

2.Left face due to the x component velocity

3.Bottom face due to the z component velocity



x component momentum balance,

$$d\vec{m}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=dF_x=dF_{Bx}+dF_{Sx}$$

Body force acting on the control volume will be the gravity and it is acting only in one direction as per the figure and that is in the negative direction of z. In all other direction, the value of body force will be zero. Now, we are looking into how the x component velocity causes stress or surface force to the control volume.

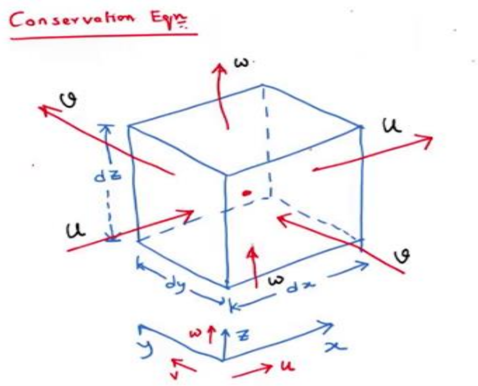
This control volume comprises of 3 types of surfaces(6faces):

1.yz plane (left and right face as per the figure)

2.xy plane (bottom and top face as per the figure)

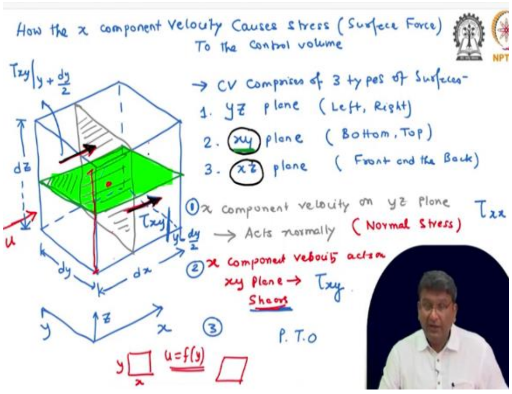
3.xz plane (Front and back face as per the figure)

Here we are considering all these three surfaces are passing through the centroid: yz plane that is passing through the centroid and x y plane that is passing through the centroid and xz plane passing through the centroid.

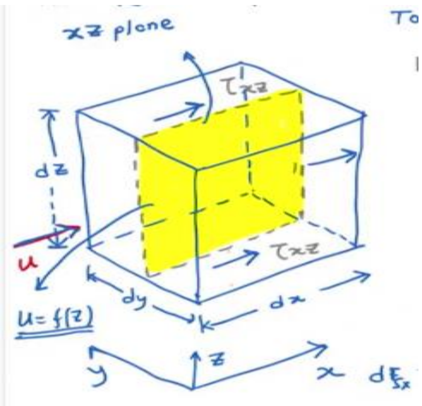


If we are considering the yz plane passing through the centroid, x component velocity acts normally on the yz plane (both left and right face of the control volume) and gives normal stress. This x component velocity generates upstream flow in the left face of the control volume, and it causes the shear stress. Because of the flow within the control volume, it applies a stress towards the right face of the control volume.τ_{xx} implies the total stress due to the x component velocity acting normally on the yz plane.

Next, we are looking at the how the x component velocity acting on the xy plane (green color), actually it shears. And it gives rises to shear stress τ_{xy}.similarly, x component velocities act on the x z plane and it causes shear stress τ_{xz}.



How the x component velocity causes stress (surface force) to the xz plane (yellow colour) control volume.



The x component velocity causes shear stress over the xz plane on the top and bottom which is τ_{xz}. Because of the flow over the control volume, it is shearing, and it is resulting in stress.

So, now we understand that the x component velocity causes 3 types of or interaction with the surface or applies force on three types of surfaces or 3 sets of surfaces on the control volume. It applies a normal stress on the on-y z plane that is the left face or a y z plane passing through the centroid, and it acts it causes shear stress on the on x y plane and on x z plane.so the net surface forces acting on the control volume,

$$dF_{sx} = \left\{\sigma_{xx}\big|_{x+\frac{dx}{2}}.dydz - \sigma_{xx}\big|_{x-\frac{dx}{2}}.dydz\right\} + \left\{\tau_{xy}\big|_{y+\frac{dy}{2}}.dxdz - \tau_{xy}\big|_{y-\frac{dy}{2}}.dxdz\right\} + \left\{\tau_{xz}\big|_{z+\frac{dz}{2}}.dxdy - \tau_{xz}\big|_{z-\frac{dz}{2}}.dxdy\right\}$$

$$dF_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) dxdydz$$

So we can write,

$$d\vec{m}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=dF_x=dF_{Bx}+dF_{Sx}$$

$$d\vec{m}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)dxdydz+\ dF_{Bx}$$

$$\rho dV\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)dxdydz$$

So the net exchange of mometum for steady state system becomes,

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)$$

This equation is known as the Navier’s equation or the Cauchy’s equation.

Note:we are assuming the system in steady state.

Thank you very much.