Chemical Engineering Fluid Dynamics and Heat Transfer
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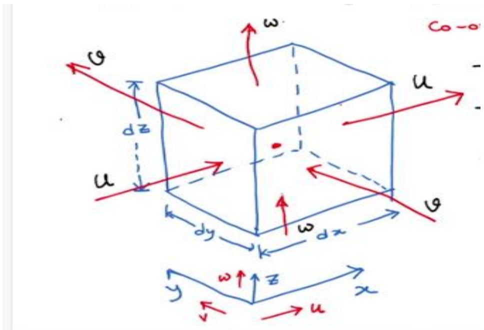
Lecture - 07
Conservation Equation 02

Welcome back to the 7th lecture. So, the previous lecture we were discussing about the Derivation of the Conservation of Mass or a Continuity Equation for the Cartesian coordinate system. Consider a control volume with the sides dx, dy and dz.



So, we now understand that there is mass entering through the left face right, due to x component velocity and leaving through the right face. And mass entering the front face due to y component velocity and leaving due to y component velocity at the back face. And similarly mass entering at the bottom face and top face due to z component velocity. We will investigate each term of the conservation equation.

So, mass entering through the left face due to the x component velocity will be $\left[(\rho u)|_{x-\frac{dx}{2}}.dydz\right]$.

Mass leaving through the right face due to the x component velocity will be $\left[(\rho u)|_{x+\frac{dx}{2}}.dydz\right]$

Therefore, Net efflux in the x direction or net exchange of mass in the x direction will be $\left[(\rho u)|_{x+\frac{dx}{2}}.dydz\right]-\left[(\rho u)|_{x-\frac{dx}{2}}.dydz\right] = \frac{\partial(\rho u)}{\partial x}.dx.dydz = \frac{\partial(\rho u)}{\partial x}.dV$

We know that the volume of the control volume, $V = dxdydz$

So, the net efflux in the x direction $= \frac{\partial(\rho u)}{\partial x} dV$

Similarly, the net efflux in y direction $= \frac{\partial(\rho v)}{\partial y} dV$

The net efflux in z direction = $\frac{\partial(\rho w)}{\partial z} . dV$

Accumulation term = $\frac{\partial(\rho V)}{\partial t}$

since V is constant, Mass accumulated within the control volume = V $\frac{\partial \rho}{\partial t}$

There is no possibility of any generation in this case, therefore mass cannot be created or destroyed. If we are putting all the terms in the general mass conservation, equation becomes

$$\frac{\partial(\rho u)}{\partial x} dV + \frac{\partial(\rho v)}{\partial y} dV + \frac{\partial(\rho w)}{\partial z} dV = dV \frac{\partial \rho}{\partial t}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \frac{\partial \rho}{\partial t},$$

this is the generic form of the continuity equation in the differential form for any fluid.

For an incompressible flow, $\rho \neq \rho(x, y, z, t)$, ρ is constant,

$$\frac{\partial \rho}{\partial t} = 0$$

Then the above equation becomes, $\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$

$\rho \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0 \Longrightarrow$ continuity equation valid for an incompressible fluid.

Note: The above continuity equation is valid for both steady and unsteady state flow of incompressible fluid. The continuity equation can be written in terms of the linear deformation, then the equation becomes

$$\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} + \dot{\epsilon}_{zz} = 0$$

Stream Function:

Unlike streamlines, stream function is mainly defined for 2D flow. It is a point function $\psi(x, y, t)$ that is defined as

$$u = \frac{\partial \psi}{\partial y} : v = -\frac{\partial \psi}{\partial x}$$

For continuity equation of 2D flow,

$$\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0$$

If we plug the value of u and v in the continuity equation, we will get

$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$: It implies that stream function satisfies the continuity equation.

By the definition of vorticity, $\omega_z = \frac{1}{2}(\dot{\alpha} - \dot{\beta}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

If the flow is irrational flow, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, vorticity $\omega_z = 0$,

therefore, the above equation becomes $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

if we substitute u and v in terms of ψ, then the equation becomes

$$-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0: \quad \nabla^2 \psi = 0: \quad it \text{ is valid only for an irrotaional flow.}$$

To check whether the flow is irrotational or not, check the Laplacian of the stream function. If it is zero, then the flow will be irrotational flow. From the previous lecture we understood that at any instance of time, the tangent to a streamline gives the direction of the velocity. By following an Eulerian approach there will be infinite number of streamlines within a flow field. So how will you identify the streamlines?

Along a streamline the value of stream function is constant. ψ is a point function of x, y for a steady state flow.

$$\psi = f(x, y)$$

if we take the differential of the above equation

$$d\psi = \frac{\partial \psi}{\partial x}.dx + \frac{\partial \psi}{\partial y}.dy$$

From the definition of stream function, we know that

$$u = \frac{\partial \psi}{\partial y} : v = -\frac{\partial \psi}{\partial x}$$

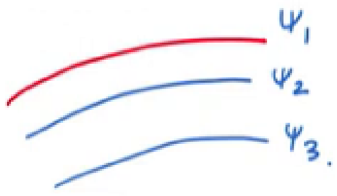
So we will get

$$d\psi = -vdx + udy$$

By the equation of streamline, $\frac{u}{dx} = \frac{v}{dy}$: $udy = vdx$

If we substitute this in the above equation: $d\psi = -vdx + udy = 0$

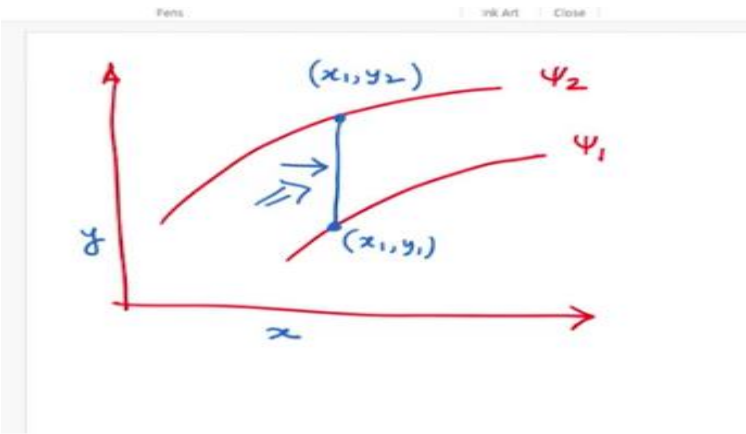
Note: That means $\psi = constant$: The value of the stream function is constant along that streamline. Each streamlines have different values of stream function.



Consider two streamlines having two different stream functions ψ_1 and ψ_2 and Q is the flow rate between the two streamlines.

Therefore, flow rate across the streamlines,

Q $= \int_{y_1}^{y_2} udy = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y}\, dy = \int_{y_1}^{y_2} d\psi = \psi_2 - \psi_1$



Simply by subtracting the value of corresponding stream functions of two streamlines we can find out the volumetric flow rate between these two streamlines.

Velocity potential:

Like the way stream function is valid only for a 2D flow field, the velocity potential is valid only for an irrotational flow.

Velocity potential is defined as $\vec{V} = -grad\ \phi$

In the scalar form we can written as $u = -\frac{\partial \phi}{\partial x}$, $v = -\frac{\partial \phi}{\partial y}$, $w = -\frac{\partial \phi}{\partial z}$

The negative sign essentially is a convention it represents that ϕ decreases in the direction of the flow, Like the concept of temperature decreasing in the direction of heat conduction.

If we consider a 2D irrotational and incompressible flow,

For irrotational flow , $u = \frac{\partial \psi}{\partial y} : v = -\frac{\partial \psi}{\partial x}$ $\Rightarrow$ $\nabla^2 \psi = 0$

From the definition of velocity potential, $u = -\frac{\partial \phi}{\partial x} : v = -\frac{\partial \phi}{\partial y}$ $\Rightarrow$ For continuity equation of 2D flow,

$$\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0$$

If we substitute for u and v in the above equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0: \quad \nabla^2 \phi = 0$$

That means, Laplacian of the stream function and the velocity potential are simultaneously zero, for an irrotational incompressible flow field. By checking these tests, you can identify whether the flow is irrotational or incompressible. In the next lecture we will discuss about the conservation of momentum.

Thank you very much.