Chemical Engineering Fluid Dynamics and Heat Transfer

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Lecture - 45
Transient Heat Conduction (Contd.)

Hello and welcome back once again with another lecture on Transient Heat Conduction in NPTEL online certification course that is on Chemical Engineering Fluid Dynamics and Heat Transfer. In the last two classes that we have discussed regarding the transient heat conduction.

The first we started with a general understanding of transient heat conduction and the lumped capacitance method. Why is it popular? The reason for its popularity was its simplicity, but then at the same time its validity has to be assessed that till how much or what is the logic behind that.

So, we have understood that logic as well. Now, the point is that when that is not valid when the Biot number is no more less than 0.1; that means, there is a spatial variation which is significant. And if we use lumped capacitance method it would give us inaccurate result. Now, in those cases we can always go for the analytical solutions of the heat equations that we have seen earlier.

In all three direction dimensions we have seen the heat equation at the very beginning of the conduction where we had the transient term all the terms were there in three dimension. There was heat generation term, there was transient term. So, that is complicated and each of their solution that involves appropriate boundary condition.

Because we had restricted our discussion till this point is for the one dimension. And in fact, we will continue with the one dimension for its simplicity, but these ideas can be easily extrapolated to multiple direction dimension. So, this in the case of spatial distribution along with temporal distribution we have to understand some methods that are numerically in nature along because analytical solutions are not trivial in those cases.

It is there in several text books standard state book you can find out the solution of the heat equation with appropriate boundary condition, but those standard formats are given for a set of boundary conditions that can be there. Now, what I will show you here we will

demonstrate a numerical method that is called the finite difference method many of you have already been through because of your mathematics classes.

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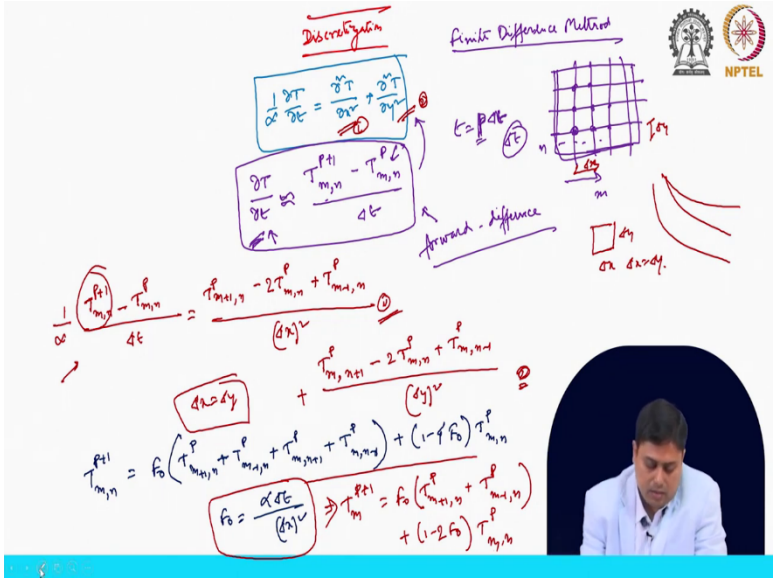

So, here we use a term called the discretization. We call the discretization. So, discretization essentially means converting partial differential equation to a set of algebraic equation which is easier to solve.

So, for example, in two dimension the heat equation that we have seen a full-fledged heat equation without any heat generation we can write as:

$$\frac{1}{\propto}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

the problem can be complicated further by adding heat generation term we are say not looking into that.

But even this equation solving analytically for various boundary conditions are difficult it can be done. And that actually necessitates the understanding of numerical methods. So, one of the methods I will describe it here very briefly , I hope that would be helpful for you. Now, in this case the process that will or the method that will use is called the finite difference method. Now, what happens in the numerical methods? We go step by step or we march step by step.

So, for this what happens we usually divide the domain or the object the that we say the computational domain into several number of meshes or cells. The understanding becomes simpler if the cell sizes are uniform, but it necessarily that need not be the meshes can be non-uniform in that case the again the procedure becomes convoluted. So, consider there are the object shape is such that it can be divided into small equally spaced small-small pieces or the meshes.

Now, what we will write, we will write the governing equations or we will discretize this partial differential equation for each of these cells as we go or as we march in direction as with time. So, when we jump from initial time to the final time in numerical process, we do not jump directly from the starting point to the end point we go in small steps. So, the solution accuracy improves or we get an improved solution accuracy. Now, the time interval total time is then divided in P number of small part or small time which is ΔT.

Now, in finite difference method what we do is we approximate the derivatives. Now, here we approximate since it is an approximation, I sign it accordingly and I write this as:

$$\frac{\partial T}{\partial t} \approx \frac{T_{m,n}^{P+1} - T_{m,n}^{P}}{\Delta T}$$

This is an approximate form of this term where what we have the m and n the subscripts are designated to mention a coordinate m in the x-direction n in the y-direction.

For example, if this is my m, n the next one I can consider as (m+1, n), (m + 2, n), (m + 3, n). Similarly, here if this is my (m, n) point this becomes (n + 1, m) , (n + 2, m), (n + 3,m) like this. So, this designates the coordinates. And superscript P is the number of time step that we have considered. So, here if we see that this differential part, we have discretized in two terms where we actually if we are at point P. So, (P + 1) is a new time step where we are going where we are calculating the value of the temperature at that time step. That means, P is my previous time step. At previous time what was the temperature based on which I am going to the next step which is (P + 1) with the time increment of ΔT. So, that means, we are going discretely from one point to the other point.

So, this discretization helps in simplifying this partial derivative terms in order to get simple algebraic equations that we can solve by any iterative process. Because for each and every nodal point these are the points that we say the nodal points like this is for (m,

n). Similarly, for (m + 1, n) , (m + 2, n) and (m, n + 1) , (m, n + 2) for each and every point we can write such expression. And we replace here and then find out a generic expression and then we iteratively solve that expression with time and with space.

So, now these kinds of discretization in this case that we have used since it involves a next time step value, we say that this is the forward differencing approximation or forward difference approximation. So, forward difference approximation to the time derivative in this case.

Similarly, we can use it for the spatial derivatives as well. So, how it would look like if I try if I write it here, if you follow each and every expression, I am pretty sure that you would understand that this is the first part.

$$\frac{1}{\propto}\frac{T_{m,n}^{P+1}-T_{m,n}^{P}}{\Delta T}=\frac{T_{m+1,n}^{P}-2T_{m,n}^{P}+T_{m-1,n}^{P}}{(\Delta x^2)}+\frac{T_{m,n+1}^{P}-2T_{m,n}^{P}+T_{m,n-1}^{P}}{(\Delta y^2)}$$

Δx & Δy are grid size. When we march in x-direction it is (m + 1,n) , (m – 1,n) and the middle point (m, n). And here when we march in y-direction it is (m, n + 1), (m, n – 1) and (m, n) at a particular instead in time. So, this is forward differencing with respect to time and this we have done in case of as a central differencing in space.

Now, if we simplify this and solve for this unknown because we need to know what would be the temperature at the next time step that is the goal. How the temperature profile changes with time. So, in that case what we write or if we simplify you would realize that the:

$$T_{m,n}^{P+1}=Fo\left(T_{m+1,n}^{P}+T_{m-1,n}^{P}+T_{m,n+1}^{P}+T_{m,n-1}^{P}\right)+(1-4Fo)T_{m,n}^{P}$$

where again the Fourier number is the dimensionless time and it is given by:

$$Fo=\frac{\alpha\Delta T}{(\Delta x^2)}$$

Again, the assumption here is that it is the cells are uniform that is ($\Delta x=\Delta y$). The cells are uniform in size. Once we do that what it shows this expression can further be written as:

$$T_{m}^{P+1}=Fo(T_{m+1,n}^{P}+T_{m-1,n}^{P})+(1-2Fo)T_{m}^{P}$$

T_m^{P+1} this expression now, we are simplifying instead of two dimension for one dimension. As let us say we if we restrict our discussion for the time being. Because now we can see this can be implemented for three dimension as well considering another variable another variable like m and n to define the coordinate. So, in this case what will happen for one dimensional case that is in the x direction if we consider. And particularly here since it is one dimension, we can erase this n terms because now we are not considering any n here it is simply all the m terms.

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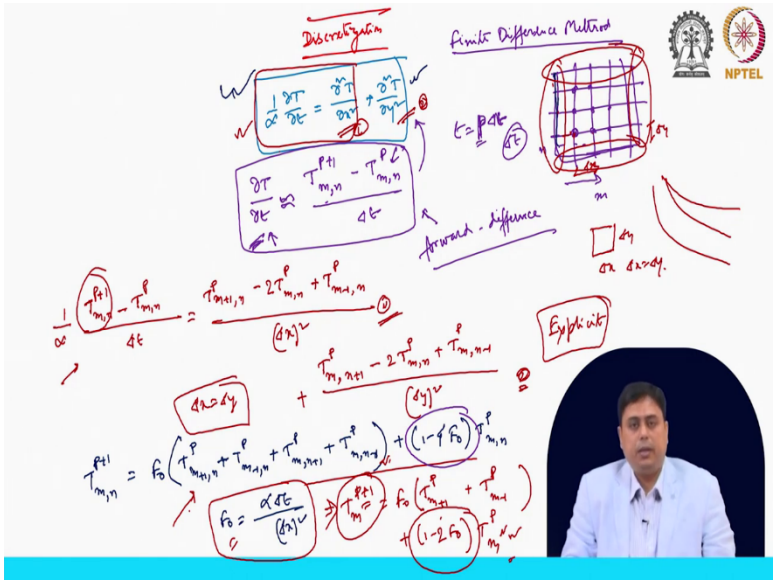

So, what it shows that the calculation of the next step temperature at any given point or at a particular point in x- direction only depends on the temperature of the previous time step.

So, which means this calculation we can do individually or independently irrespective of the information of other nodal temperature because there is no involvement of any other points at any next time step. So, this involves the information of the previous time step temperature which is usually known because even if we start a solution the previous temperature is essentially the initial temperature. So, at (t=0) either we initialize the domain with a given temperature or we assume certain value.

So, initial condition is usually known or if not known we guess it and we apply to all the nodal points. Based on that initial point we go for the next time step. And this next time step calculation does not involve any further complication, it involves only the point that is known previously.

So, this kind of equations can be solved independently and that is why such formulation is called the explicit formulation of the problem. It is easier to implement and understand or to write because of this forward differencing discretization of the time derivative. It gives a solution because all unknown nodal temperature for the new time are determined specifically by the known nodal point temperatures of the previous time.

So, quite naturally what we see that this solution of by this process of this differential equation now we have reduced it to one dimension, we can do it for any given boundary condition. Now, the point is that the accuracy of the solution improves if we find a desired Fourier number or a more accurate Fourier number and this actually dependent on the time step as well as the grid size.

The time step and the grid size; that means, this spacing in space the spatial distance between the nodal points and in time. So, depending on the values of ΔT & Δx we can find an optimum value of Fourier number for which the solution can give us our desired accuracy of the result.

Because this explicit formulation is easy to implement, but it comes with a natural influence, that it is not stable. Because stability analysis would shows that in order to be this equation have a stable result this parameter the coefficient of the time of the previous time this parameter has to be positive.

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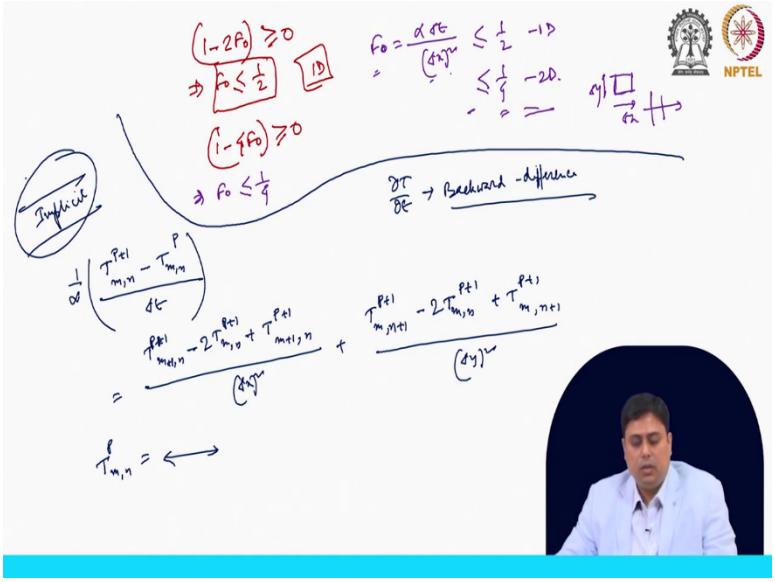

So, which means in the case that we see the stability requirement of this solution is that:

$$(1 - 2F_o) \geq 0$$

$$F_o \leq \frac{1}{2}$$

That means Fourier number has to be lesser than half in case of 1D transient heat conduction problem, if we try to find out its temporal variation. Similarly, for two dimensional cases the criteria:

$$(1 - 4F_o) \geq 0$$

$$F_o \leq \frac{1}{4}$$

And by doing so by restricting the criteria what we have here? It must be less than (1/4). And this F_o that we have already seen:

$$Fo = \frac{\alpha \Delta T}{(\Delta x^2)} \leq \frac{1}{2} \ (1D)$$

$$\leq \frac{1}{4} \ (2D)$$

Above is the criteria to have a stable solution by explicit formulation.

And accordingly, then you can choose the ΔT & Δx value. That has to be because once you decrease the time step, once you decrease the time step the Fourier number decreases. You are in the safer region for a particular Δx. But if you march in time with a very large time step then the chances of Fourier number getting higher would be much larger.

That in those in that case the chances of unstable simulation of the solutions would be much higher. And in this case in the case of 2D this value has to be restricted beyond 0.4 or below 0.4. Now, for a particular time step if you want to have a higher time step or larger time step value accordingly you have to adjust the grid size the Δx the division in this x-direction for the case of one dimensional case and for the case of two dimensional case Δx & Δy you need to adjust accordingly.

So, that the value of Fourier number falls below to this one. So, this is the pros and cons of the explicit formulation and similarly there is implicit formulation. Implicit formulation in that case what happens? The time derivative is discretized in backward differencing. Using the backward differencing process. And if we do so let me write that expression then in that case for the implicit formulation where we need not worry about this stability criteria. The form it would look like is that:

$$\frac{1}{\alpha}\left(\frac{T_{m,n}^{P+1}+T_{m,n}^{P}}{\Delta T}\right)=\frac{T_{m+1,n}^{P+1}-2T_{m,n}^{P+1}+T_{m-1,n}^{P+1}}{(\Delta x^2)}+\frac{T_{m,n+1}^{P+1}-2T_{m,n}^{P+1}+T_{m,n-1}^{P+1}}{(\Delta y^2)}$$

Again, we can rearrange it and find an information for $T_{m,n}^{P}$ a term that would consist of all the next time step values.

So, the new temperature of point (m, n) node would depend on the new temperature of the adjacent nodes and that is why it is called the implicit formulation because you cannot solve it independently as we have done in case of explicit simulations with in marching order.

So, essentially the crux of this formulation is that we take the governing equation which is the heat equation; we take appropriate boundary condition because in order to solve this problem or solve this expression what we need is initial condition. That means, one condition with respect to time it is first order derivative with respect to time and second order derivative with respect to space or x or y.

So, we need one initial condition that is with respect to time and two boundary conditions in space that is at x is 0 this value at y is 0 that value or even at x is for two different x is we can have two different temperatures.

If those are given with respect to x and y both for two different values, we can solve this analytically as well as numerically like we have shown here. Because those boundary points are then here and when we approximate the values because those boundary values are either at x is 0 and x is L or y is 0 and y is L.

This boundary nodal points would come in these expressions when we calculate for the adjacent point. And the boundaries values would be incorporated in those cases. And that is how the solution or the boundary value propagates inside the domain. I hope this process

is clear because it is with an understanding that you have done or you understand this finite difference method.

I have given you a brief overview those who forgot I hope this would help you to remember or refresh your memory that what was the finite difference method and how we can apply it and what is discretization. So, with this we are covering this conduction part in this course in a very compact manner.

In the next week we will see convection or in the next class we will see that we start with the concept of convection mode of heat transfer. Till then I hope that you would rehearse this part with a given assignments and I hope it would help you to clear your understanding or whatever the concept you wanted to have to be cleared. I hope that would be fine with you. With this I thank you for your attention and we will see you in the next class.