# Chemical Engineering Fluid Dynamics and Heat Transfer

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## Lecture - 40
## One-dimensional Heat Conduction (Contd.)

Hello everyone. Welcome back once again with another lecture of Heat Transfer in the NPTEL online certification course Chemical Engineering Fluid Dynamics and Heat Transfer.

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The thing that we were discussing is One-dimensional Heat Conduction and particularly in the radial coordinate or say the cylindrical coordinate specific focus on the cylindrical coordinate.

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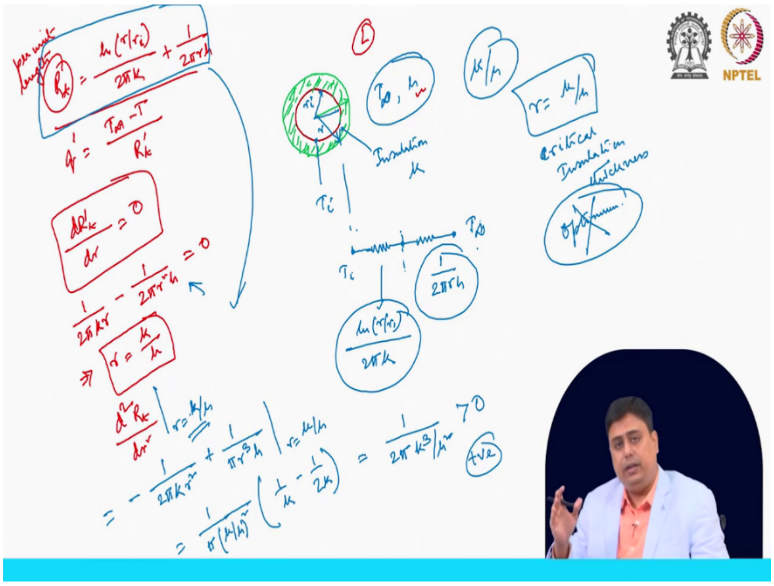


And in the last class we tried to find out a unique system that in such a in this system which is shown on the slide is that say for example, we have a thin wall tube containing a fluid flowing inside it. Now, thin wall means the assumption that we have made is that the wall this material does not have in thermal resistance.

So, it attains the fluid temperature uniformly and this is wrapped by an insulator in order to minimize the heat loss. Now, the point that you must have now noticed is that with increasing the thickness of the wall or say the composite wall the resistance to conduction increases or the conduction resistances increases.

On the other hand, as you increasing the outer surface area the convection heat transfer increases, which means while you are trying to reduce the conduction heat transfer the heat transfer by convection increases. So; that means, the logical thing that comes to our mind is that there must exist a certain thickness which balances these two.

That means we will try to find out whether there exist any optimum insulation thickness for this cylinder, for the tube that contains a fluid having a different temperature than its surrounding. So, this was the problem statement and in order to do so as I have mentioned earlier, we made couple of assumptions that this is a steady state scenario.

The material resistance, the wall resistance, the thermal resistance is negligible and in this case the radiation from the insulation outer surface to the surrounding is also negligible.

And we have drawn a equivalent thermal circuit for this system which is here. Now, this means that the total resistance if we can write the R total is essentially:

$$\frac{\ln\left(\frac{r}{r_i}\right)}{2\pi k}+\frac{1}{2\pi rh}$$

This is the total thermal resistance per unit length of the tube. So, that means the heat transfer rate that we can write per unit length of the tube let say define it by:

$$q' = \frac{T_\infty - T}{R_t}$$

Let us also designate this as $R''$ because it is per unit length the total resistance per unit length.

So, if there exists an optimum thickness that would be; that means, it is associated with the value that would be with R that is what we are trying to find out that what would be the optimum value of R for which this resistance conduction resistance that increases; that means, that reduces the conduction heat transfer rate.

But on the other side it increases the convection heat transfer rate that balances. So; that means, what we can do we can find out this value with respect to R we do this; that means, this such a optimum value can be attained if we calculate this or if we estimate this parameter.

$$\frac{dR_t'}{dr} = 0$$

So, we take a derivative of this which comes out to be from here it comes out to be:

$$\frac{1}{2\pi kr}-\frac{1}{2\pi r^2 h} = 0$$

$$r = \frac{k}{h}$$

This has a tremendous significance the r that we are trying to find out an optimum value of r is equals to the thermal conductivity of the insulator or the material by which we are insulating divided by the outer fluid convection heat transfer coefficient.

Now, whether this value now leads to the maximum or it actually maximizes or minimizes the total resistance we can find out that if only we find out the double derivative of that this value, this resistance. This parameter now we have to calculate or we have to estimate.

$$\frac{d^2R_t}{dr^2} = 0$$

$$= -\frac{1}{2\pi k r^2} + \frac{1}{\pi r^3 h}$$

$$\left.\frac{d^2R_t}{dr^2}\right|_{r=\frac{k}{h}} = \frac{1}{\pi(\frac{k}{h})^2}\left(\frac{1}{k} - \frac{1}{2k}\right)$$

$$\frac{1}{2\pi\frac{k^3}{h^2}} > 0$$

That means this double derivative results in always positive value for $r = \frac{k}{h}$; that means, till this point that the numerical value when we calculate, $\frac{k}{h}$ till this numeric value of r that much insulation would cause the this much insulation radius would cause the total resistance to be a minimum value to be a minimum; that means, and this much resistance is not a maximum resistance value.

If we put insulation of $r = \frac{k}{h}$ it actually minimizes the total resistance of the system; that means, you cannot find a optimum value, it is not an optimum insulation thickness when we put on this insulator around a cylinder. So, optimum insulation thickness does not exist what exists is the critical insulation thickness.

That means, this much minimum is necessary because if this is the minimum resistance the heat transfer rate is maximum at this position. On top of which whatever you add as the insulation it will from there onward it will reduce the heat transfer rate.

So, this much insulation is minimum where minimum necessity and then the optimum value if somebody tries to find out that optimum value depends on the cost analysis of the insulator how much thickness you can give as well as the space constraint or the design parameter because in certain cases you cannot put on infinite insulation, the cost increases as well as sometimes space is not there in order to put that much insulation.

So, the point is the term is $r = \frac{k}{h}$ is called the critical insulation thickness and not optimum it is not optimum, because it maximizes heat transfer that is below which the heat transfer rate increases with increasing r above which heat transfer rate decreases with increasing the value of r.

So, this is a very vital concept that is necessary to know because based on this several numerical problems can be formed or this can be used for in order to find out a critical insulation thickness of a pipe flowing a material or a fluid that is having a different temperature than its surrounding.

The typical question comes how much insulation we need to provide so that the fluid temperature can be retained or the heat loss can be minimized. So, what you can find out is a critical value that is necessary and beyond it that is the value you have to use based on the cost analysis or the space constraint. So, I hope this concept is clear that why it is critical and it is not an optimum value. Now, similar concept but in a different methodology, that we can apply for sphere.

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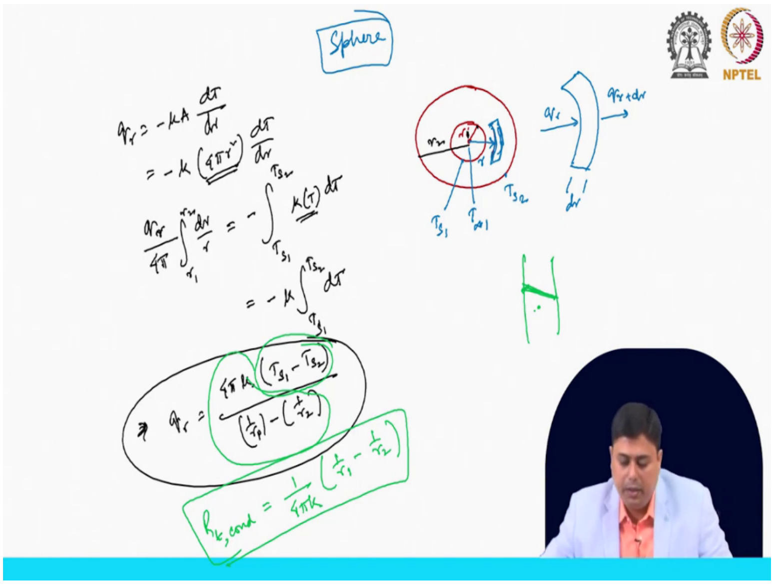


So, for sphere if we try to apply the similar concept say instead of now cylinder. So, this is the $r_i$ and now here instead of the methodology that we have used there is an alternate way to find out this equivalent thermal circuit or say the resistance values that we are now showing is by say elemental analysis the differential elemental analysis.

So, which is say we took a small element that is at a distance radial distance r. So, say the inner fluid temperature that we have is $T_{\infty 1}$ and this is $T_{S2}$ because usually that we consider this as the $T_{S1}$. Now, for this element that we have considered, so this element that we are considering if I redraw that part only where say this is my $q_r$ a differential element of thickness dr. So, this is $q_{r+dr}$.

So, conduction in a spherical shell you can consider. So, now here a steady state one-dimensional analysis with no heat generation what we can write for such case is that a simple expression that we can write down is basically:

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

Where, this A is $(4\pi r^2)$ that is the area normal to the direction of heat transfer in this case.

So, what we see that $q_r$ is a constant and it is independent of r from this expression because this is the area across which this is happening. Now, if we try if we integrate it for this differential element that is:

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r} = -\int_{T_{S1}}^{T_{S2}} k(T)dT = -k\int_{T_{S1}}^{T_{S2}} dT$$

$$q_r = \frac{4\pi k(T_{S1} - T_{S2})}{\left(\frac{1}{r_1}\right) - \left(\frac{1}{r_2}\right)}$$

assuming k as constant.

So, again if we try to find out an expression of resistance, if we try to find out an expression of resistance, we can clearly identify that our driving force is this part $\Delta T$. So, the rest of the parameter is our resistance which would be in inverse to it. So, which means R total for this system the conduction resistance is essentially:

$$R_{t,cond} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

So, temperature distribution also can be obtained by the standard process that we have followed earlier, but here I have just shown you how to calculate this or how to find out the resistance in the case of a sphere. So, now what we will do we will quickly introduce the section that contains an interesting part which we call this as the extended surface.

But before that we will go into a part which because all these discussions, we have done is based on the assumptions that there is no heat generation or no heat source or sink term. And that is why in Cartesian coordinate we have seen the temperature profile is linear that we derived inside a domain or inside a medium. The temperature profile due to conduction is linear value.

Now, the interesting part is that there can be a heat source term or constant heat generation. The reason we already discussed that say inside that material some chemical reaction is happening which is exothermic in nature, that chemical energy is converted to the heat energy thermal energy. And that dissipates and it constantly say generating that amount of heat, so in such case what would be the temperature profile or the temperature distribution inside the medium.

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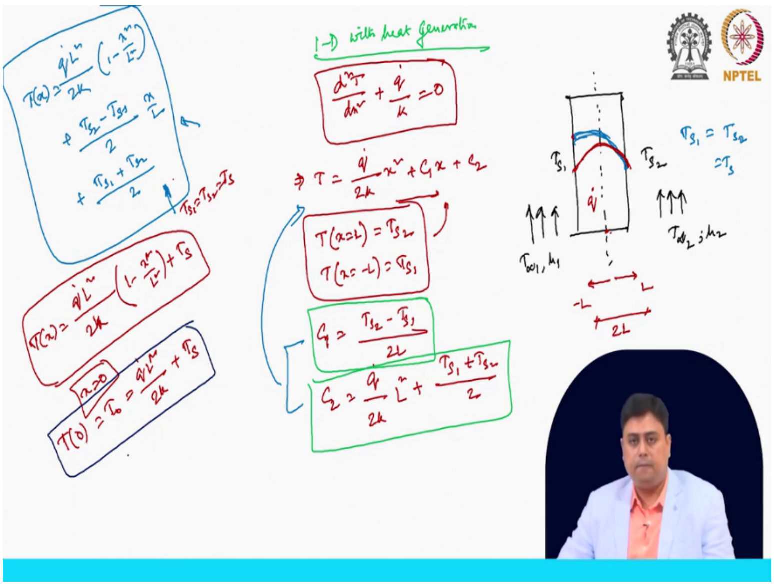


So, now the case begins is that the 1-D heat transfer with heat generation. Again, if we go back to the Cartesian coordinate and we write the governing equation, but still remaining in the one-dimensional part what we can write the heat diffusion equation then becomes:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

This is our heat diffusion equation in one-dimension with heat generation at steady state where $\dot{q}$ is the volumetric heat generation term.

Again, similar to the procedure that we followed earlier, if we try to find out its solution analytically its generic solution is that:

$$T = \frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

where $C_1$ and $C_2$ are the integration constant of the constant of integration. Now, in order to solve it or find a unique solution we have to know what are the boundary condition.

We need two boundary condition with respect to x. Now, usually what happens as we have seen earlier for a wall or slab say this is $T_{S1}$ and this is $T_{S2}$. A fluid is flowing outside or on both the sides with two different temperature and two different fluid. This is the plane of symmetry.

Now, this $\dot{q}$ is in the medium itself either by wrapping an electrical coil on it and passing the electrical energy constantly or some means of chemical energy being dissipated into thermal energy. Now, what would be in this case the temperature profile say if $T_{S1}$ is higher than $T_{S2}$ in absence of $\dot{q}$ it was linear. But here from this expression we can see this in this case that would not be the case.

So, if we apply a simple at (x = L) say we have or say based on the plane of symmetry. Now, if we designate this as towards x direction the positive direction it is having L and here it is having (-L); that means, it is having a 2L thickness. Since, we are defining now this as a plane of symmetry.

$$T(x = L) = T_{S2}$$

$$T(x = -L) = T_{S1}$$

If these two temperatures are known to us; that means, these are the two boundary conditions that are given we can replace it here in order to find out a generic expression.

$$C_1 = \frac{T_{S2} - T_{S1}}{2L}$$

$$C_2 = \frac{\dot{q}}{2k}L^2 + \frac{T_{S2} - T_{S1}}{2}$$

The constant will have these two forms for this generic formulation.

So, which means now if we replace this the temperature distribution so if we now replace this two parameter here in on this expression T what we find is that:

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{S2} - T_{S1}}{2}\frac{x}{L} + \frac{T_{S2} + T_{S1}}{2}$$

Once we find the temperature distribution, we can always find out the heat flux from the Fourier's law. Now, here what we can see that with heat generation the heat flux would no longer be independent of x. The heat flux would also vary along the direction of x. So, now for the sake of simplicity or in a specific scenario that also happens many a time that if , $T_{S1} = T_{S2} = T_S$

So, both the sides are at a uniform temperature or the same temperature and there is a heat generation term. In that case this expression is simplified and it gives a symmetrical form because then if this temperature is not further higher than this one. Because here now if we have a temperature which is not giving up linear profile based on this expression that we see and that profile may look like this kind of a profile.

But when this $T_{S1} = T_{S2}$ it gives us a symmetric profile like this and that expressions, we can easily find:

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_S$$

$$T(x = 0) = \frac{\dot{q}L^2}{2k} + T_S$$

So, this gives us the temperature distribution when there are two cases that we have discussed when $T_{S1}$ is not equals to $T_{S2}$ which is a generic formulation and in specific case when $T_{S1}$ or say the both sides of this wall are at a identical temperature with heat generation inside the medium. Then what can be the temperature profile which is the perfectly parabolic giving a symmetry on giving a plane of symmetry and the maximum temperature is attained at the midpoint of the domain.

So, we will see other applications of this concept in the fourth coming classes and we will see another interesting part in this one-dimensional heat conduction that is the extended surface, but that is in the next class. Till then I want you to go through this material understand the part that I have mentioned here also you can refer to the textbook that I have mentioned in the in my first class and I hope you would be able to understand most of the things at ease.

With this I thank you for your attention and we will see you in the next class.