Chemical Engineering Fluid Dynamics and Heat Transfer
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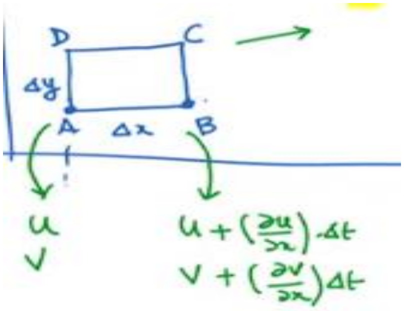
Lecture - 04
Kinematics - 04

So, welcome back. We started to discuss the deformation of fluid particles, and we have looked into the cases of pure translation and translation with linear deformation, continuous linear deformation. In this lecture we are going to talk about a more complex situation which is essentially translation with linear and angular deformation.

In the previous case, we have talked of a situation where u was a function of x only and v was a function of y only. Here we are assuming a 2D fluid element with time independent flow. So, if we are considering the same fluid particle ABCD, with u is a function of x and y, u = f (x, y), the point A is moving with velocity u and v. and v is also a function of x and y, v= f (x, y), therefore, point B has not only higher x component velocity, but also higher y component velocity.



Since B has a higher x component velocity, the length of segment AB increases as time progresses. Even though B has a higher y component velocity compared to point A, the line starts to rotate or does not remain horizontal anymore. It starts to rotate with an angle with respect to the horizontal.

So, if we are looking into the shape of the particle after time Δt,

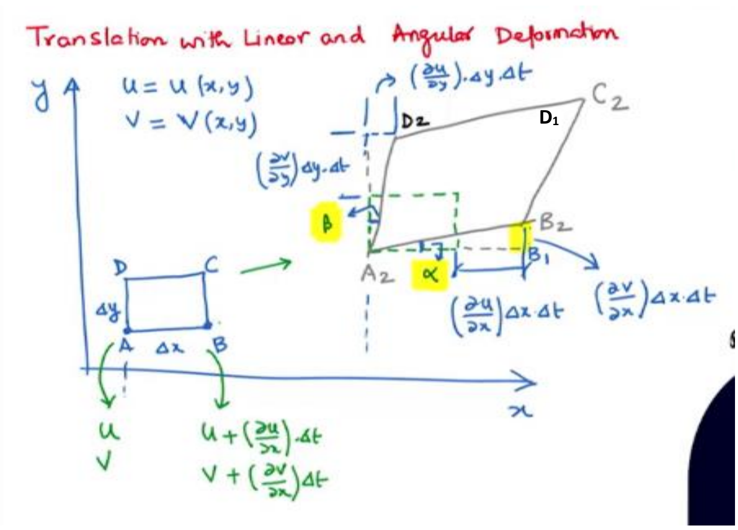
x component velocity at the point A = u

x component velocity at the point B = $u + \left(\frac{\partial u}{\partial x}\right).\Delta t$

y component velocity at the point A = v

y component velocity at the point B = v+ $\left(\frac{\partial v}{\partial x}\right).\Delta t$

Because of u as a function of x, the line segment AB becomes longer and v is also a function of x, the line segment AB tilts and results into A_2B_2. Similarly, this same happens to line segment AD. Since v is a function of y it extends, and because of u is a function of y, it tilts. And you get a shape of the particle which is like this.



The distance between point A and point A_2, l_x = u d x.

Distance between point B and point B_1 = $\left(\frac{\partial u}{\partial x}\right).\Delta x.\Delta t$

The length of line segment B_1B_2 = $\left(\frac{\partial v}{\partial x}\right).\Delta x.\Delta t$

The distance from B to B 1 is v multiplied by the distance in the y direction. B, i.e. (v.Δt). And now the additional amount by which the point has gone is basically due to variation of v as a function of x.

Similarly, distance between point D and point D_1 = $\left(\frac{\partial v}{\partial y}\right).\Delta y.\Delta t$

The length of the line segment, D_1D_2 = $\left(\frac{\partial u}{\partial y}\right).\Delta y.\Delta t$

Consider that α is the angle that line segment A_2B_2 makes with the horizontal and β is the angle that the line segment A_2D_2 makes with the vertical direction.so the total angular deformation will be the sum of these two angles.

Total angular deformation, $\gamma_{xy} = \alpha + \beta$

Initially the angle between the line segments AB and AD = 90°

After the deformation the angle between the line segments A_2B_2 and A_2D_2 reduces. And it becomes, angle between A_2B_2 and A_2D_2 = $90 - \alpha - \beta$.

If we are taking tan of angle, α,we will get:

$$\tan\alpha = \frac{\left(\frac{\partial v}{\partial x}\right).\Delta x.\Delta t}{\Delta x+\left(\frac{\partial u}{\partial x}\right).\Delta x.\Delta t}$$

$$= \frac{\left(\frac{\partial v}{\partial x}\right).\Delta t}{1+\left(\frac{\partial u}{\partial x}\right).\Delta t}$$

Here we are considering the first assumption that: $\boldsymbol{\left(\frac{\partial u}{\partial x}\right).\Delta t \ll 1}$

Therefore, $\tan\alpha = (\frac{\partial v}{\partial x}).\Delta t$

Next assumption we are considering that: for small value of α , **tan $\boldsymbol{\alpha \cong \alpha}$**

Therefore,for small value of $\alpha = (\frac{\partial v}{\partial x}).\Delta t$

$\frac{\alpha}{\Delta t} = (\frac{\partial v}{\partial x})$ This is the rate of change of α.

that means, $\dot{\alpha} = (\frac{\partial v}{\partial x})$

Similarly, it can be shown that rate of change of β ,

$$\dot{\beta} = (\frac{\partial u}{\partial y})$$

The total angular deformation will keep on increasing or changing as a function of time. Therefore, rate of angular deformation,

$$\dot{\gamma_{xy}} = \frac{\gamma_{xy}}{\Delta t} = (\frac{\partial u}{\partial y})+(\frac{\partial v}{\partial x})$$

In this example, $\dot{\gamma_{xy}}$ means total deformation or the rate of change in the included angle of the 2D fluid element or particle which is resting on the xy plane. It can be defined as the combined transverse displacement of point B with respect to A and lateral displacement of point D with respect to A.

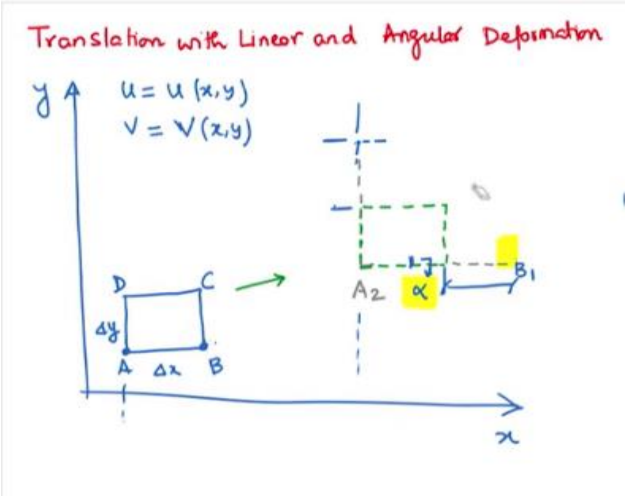
For the case of rotation, we have to consider the direction of rotation. That means, the line segment AB has rotated by an angle α in anticlockwise direction and deformed into the line segment A_2B_2 and the line segment AD has rotated by an angle β in clockwise direction, becomes A_2D_2.If we are considering the sign, then the rotation of line segment AB will be positive and the rotation of the line segment AD will be negative. Therefore, rotation can be defined as

For a 2D fluid particle on xy plane, then the rotation about Z axis will be defined as,

$$\omega_z = \frac{1}{2}(\dot{\alpha} - \dot{\beta}) = \frac{1}{2}\ (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$$

The subscript 'z' represents the axis about which the rotation occurs, or that is the line perpendicular or orthogonal to the plane. In this example, deformation is happening to the fluid element on the xy plane, and the z axis is the line perpendicular or orthogonal to the xy plane. So, then the rotation will be half of the total deformations or the change in the angle.

So here we can conclude, that if we are considering the deformation of 2D fluid element, where u and v are both functions of x and y, there will be possible chances of simultaneous angular deformation and rotation. Angular deformation is happening due to the cross functionality, i.e., variation of u with respect to y and v with respect to x.

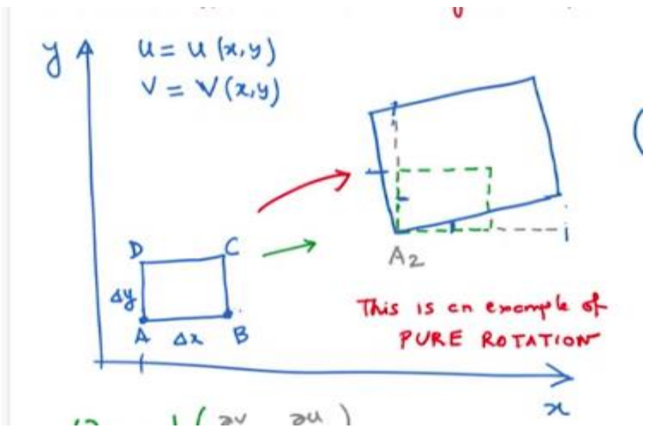


Consider a fluid element which is in a flow field, where u is a function of x and y, and v is also a function of x and y.

Case 1: $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

In this case angle α, β are numerically equal with opposite magnitude and it results that the rotation of line segment AB and AD will be in the same direction. Since there is no angular deformation, the deformation happens to be pure rotation.

Then, $\dot{\gamma_{xy}} = (\frac{\partial u}{\partial y})+(\frac{\partial v}{\partial x}) = 0$ → there is no angular deformation.that means this is the condition for pure rotation.



We have a fluid element which is in a flow field, where u and v both are function of x and y. For this particular condition though there is dilation since u is a function of x and v is a function of y, and there will be some movement of the lines, the lines do not remain horizontal or vertical. But whatever is that deformation, it cannot be captured by angular deformation only. So, it happens to be pure rotation. And in this case the angle between AB and AD remains unaltered.

Case 2: If, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$

Then the value of $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$ → Amount of rotation will be zero and this is the condition for the **Irrotational flow.**

If we are considering fluid element in 3D flow field, then.

$$\left.\begin{aligned} \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \qquad \overrightarrow{\omega} = \frac{1}{2}.\left(\Delta \times \vec{V}\right) = \text{Curl } \vec{V}$$

Another parameter to note,

Vorticity, $\vec{\Omega} = 2\overrightarrow{\omega} = \Delta \times \vec{V}$

In this lecture we have talked about the dependence of velocity on the spatial coordinates. What are the possible modes of movement as well as the deformation of a fluid particle? So, if you have velocities that are independent of the spatial coordinates, that is u and v are constant, then you have pure translation. If u is a function of x and v is a function of y only then you have only linear deformation. And if there is u and v both are functions of x and y, then you have linear deformation as well as well as angular deformation. And that led us to the concept of angular deformation as well as rotation and vorticity.