Chemical Engineering Fluid Dynamics and Heat Transfer

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Lecture - 34
Fundamentals and Mechanism of Heat Transfer (Contd.)

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Hello, everyone. Welcome back to the another class of Chemical Engineering Fluid Dynamics and Heat Transfer. We discussed last time about several thermo-physical properties and we started to understand the differential control volume analysis for the heat equation. So, today we will see that derivations that how this is developed for heat conduction.

Now, we will see this only for Cartesian coordinate and we will also look at the spherical as well as the cylindrical coordinates, but we will not go into the derivations in those coordinates, we will just see the equations and its form. And, also we will see it is applications in the subsequent classes.

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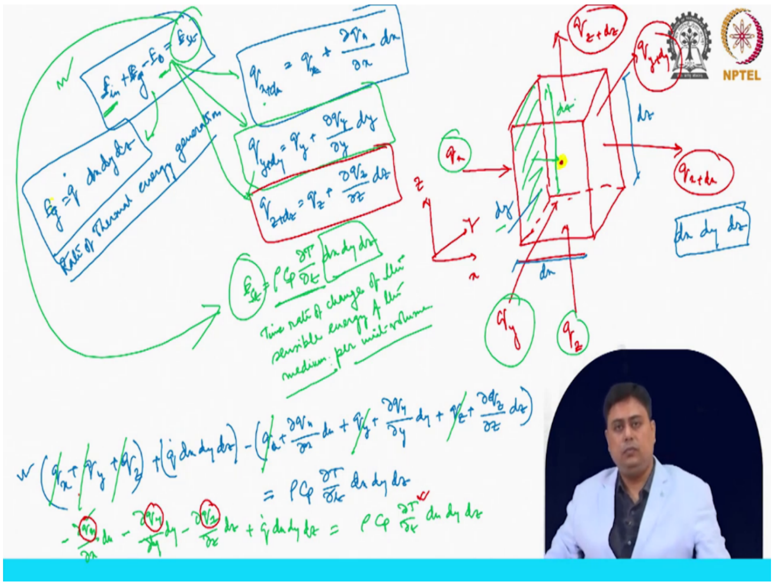

So, say we have a differential element that may look like. So, consider that this is a small differential element that in the Cartesian coordinate where we consider this as the z-direction, x and y. So, here the faces that we have, this is q_z, this is in the z-direction. So, if we consider this differential control volume of dx, dz and this we consider as dy element.

So, the control volume that we have dx, dy and dz, this is the differential volume. So, for this element now, if we consider this elemental analysis that amount of heat that is going into this element the amount of heat coming out of this element and its energy storage capacity. In presence of any energy generation term that equation that we have seen earlier what we write typically in those case that

$$E_{in} + E_g - E_{out} = E_{stored}$$

E_{stored} is the capacity to store the thermal energy. So, if we now look into this inlet, outlet or the input, output these components what we see that, what we can write for this differential elements that:

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

It's a basically Taylor series expansion where we neglect the higher order terms.

So, similarly for other elements what we can write these two terms.

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$

And, so for the three directions we can write this or expand this corresponding terms with respect to its input component. Now, if within the media, if there is any generation say consider there is there can be different modes of generation or the reason source for generation.

Now, there can be different source for this generation and if this generation happens uniformly what we can consider is that this generation term, E_g :

$$E_g = \dot{q} dxdydz$$

So, this term would be the rate of thermal energy generation.

So, what we say this as the rate of thermal energy generation. Now, this generation is different than the its capacities to store the energy because we see that the energy that goes into the system plus this generation amount and the amount it leaves the system or leaves this differential element is essentially the storage capacity of the element.

So, that the storage term how we can write? The energy storage term can be written as:

$$E_{st} = \rho c_p \frac{\partial T}{\partial t} dxdydz$$

It is a transient scenario, that means the temperature changes with time until it reaches an equilibrium.

So, here what we are considering is that the material is not experiencing any change in phase; that means, latent heat calculation is not incorporated here it is the only sensible energy or sensible temperature that is being altered ok. So, the energy storage term that we can write here is this the first three terms that is, $\rho c_p \frac{\partial T}{\partial t}$ is the time rate of change of the sensible energy of the medium having a heat capacity, ρc_p per unit volume.

And, when it is multiplied by this volume it becomes the time rate of change of sensible energy of the medium that is the energy storage. So, in this equation we know what is going in these are the terms in all the three direction the energy generation term, energy out that we have explicitly mentioned now here and energy storage term that is here.

So, now we replace all these expressions in this equation, once we do that and then we simplify it; that means, we replace all these q_{in} terms as a so, the E_{in} term that we write as:

$$\left(q_x + q_y + q_z\right) + (\dot{q}dxdydz) - \left(q_x + \frac{\partial q_x}{\partial x}dx + q_y + \frac{\partial q_y}{\partial y}dy + q_z + \frac{\partial q_z}{\partial z}dz\right) = \rho c_p \frac{\partial T}{\partial t}dxdydz$$

So, now we see once we look deep into this equation what we see is that there are several terms would be straightly straight forward manner it would be eliminated. So, for example, if we see this, we see this term this term this term goes out and corresponding this term ok. Now, also what we see that there is a common component, if we divide this by this expression which is your Δx, Δy and Δz or dx, dy and dz, we get further simplifying expression.

So, let me write it that so, here from we have:

$$-\frac{\partial q_x}{\partial x}dx - \frac{\partial q_y}{\partial y}dy - \frac{\partial q_z}{\partial z}dz + \dot{q}dxdydz = \rho c_p \frac{\partial T}{\partial t}dxdydz$$

Now, further what we see that this q_x, q_y and q_z ,these components we can write Fourier's law expression in order to convert this in terms of temperature.

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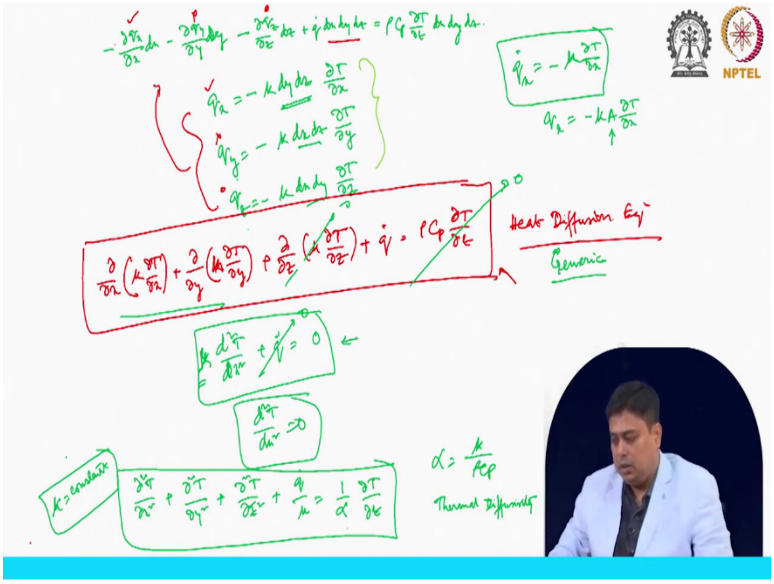

So, in this case what we do, so let me rewrite that expression once again for the sake of continuity. So, we had now this conduction this q_x is nothing, but the rate of heat conduction in x-direction that we can write in this form,

$$q_x = -kdydz\frac{\partial T}{\partial x}$$

So, this is the area that is happening here. These are the heat flux component we know that we have seen:

$$q_x^{\cdot} =- k\frac{\partial T}{\partial x}$$

When it is multiplied by the corresponding area through which this heat transfer is happening. So, A in the x direction the heat flux is going normal to the isotherm or the isothermal surface which is (dx×dz) in case of this area that you can see this is normal to that heat flux direction.

So, similarly what we can write for other two these three equations:

$$q_y = -kdxdz\frac{\partial T}{\partial y}$$

$$q_z = -kdxdy\frac{\partial T}{\partial z}$$

and once the direction changes or the component direction changes the area accordingly changes for the y-direction, we have dx, dz plane; for the z-directions, we have dx, dy plane. So, this is the z-direction and you can see that for that direction the surface is dx, dy through which the transfer is happening.

So, further what we do we replace this expression these expressions here; that means, q_x is replaced here q_y is replaced and q_z is replaced in its corresponding position. We evaluate this and we further simplify this equation by dividing both the sides this stuff that is (dxdydz) and what we get is the generic heat transfer equation for conduction in all three directions which is:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

This equation that we have derived is the general form in Cartesian coordinate for heat conduction also called the heat diffusion equation.

So, this is the heat diffusion equation or sometimes in normal form or generic form it is called the heat equation. Now, depending on the situation or a given operating conditions or a given boundary condition this equation further simplify to solve analytically or numerically whenever it is required. So, this is the basic equation to find out temperature profile in the medium or in the domain for heat conduction even including transient conduction the transient part.

So, when we talk about say a situation specific scenario, when we talk about that it is a steady state conduction if it is a steady state conduction; that means, this part of this equation is 0. If it is further steady state and one dimensional this equation then simply boils down to:

$$k\frac{d^2T}{dx^2}+\dot{q}=0$$

Now, here this there are two levels of approximation one is that the medium is isotropic; that means, the k is not varying spatially, the value of k. At the same time, it is only in one direction and the situation is steady state. So, further constructive say a situation is that one dimensional steady state isotropic media and there is no heat generation in the medium.

This equation is then further simplified to this simple expression:

$$\frac{d^2T}{dx^2} = 0$$

because there is no heat generation in such cases. If there is a situation of 2-dimensional heat conduction happening. So, depending on in which direction that is happening we typically consider the first two; that means, x and y and z is not considered.

So, that is why it is a generic or the heat diffusion equation in generic form which based on the problem or the situation we consider several level of approximation and we simplify this equation. So, now one of the popular simplified version that we sometimes refer or we work with is that.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Now, once you see this expression you realize that what are the approximations that are made here, where $\alpha = \frac{k}{\rho c_p}$,which is thermal diffusivity. So, here what we have considered is that, k is constant this is the only simplification that has been made and several times we will see that this is a logical approximations that are typically done when its variation is not clearly mentioned.

So, this is a 3-dimensional case, isotropic medium and the heat equation. So, for several problems we will use this form of this equation in order to solve related problem. So, for example, and in so, now, if we look at the Cartesian coordinate the form of it the same similar differential balance or analysis it is done for the spherical as well as the polar coordinates or say the cylindrical coordinates.

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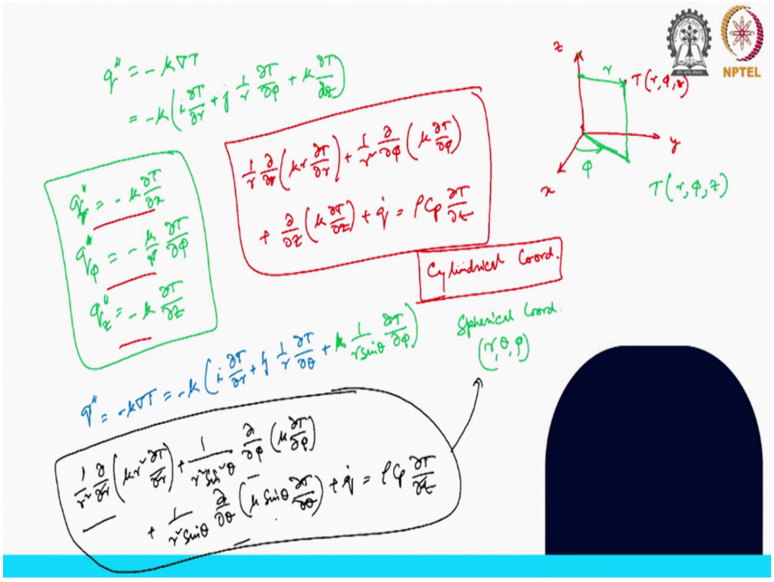

Now, if we look at the cylindrical coordinates which say we consider this as if this is the three dimensions that we are considering. Say, this is a point which is r, phi and z which where this is the z-direction. So, this is φ and this is r and this is your z-direction.

So, where T means, is the function of r, φ and z in the cylindrical coordinate in that case what we have the flux is we introduce the ∇ operator where like in this case we have seen that, it has its x-component, y-component, z-component, in the generic heat equation. In this case, we will also have it is all the three components that are:

$$q^{"} = -k\nabla T$$

$$= -k\left(i\frac{\partial T}{\partial r} + j\frac{1}{r}\frac{\partial T}{\partial \varphi} + k\frac{\partial T}{\partial z}\right)$$

So, which means it is the r component flux is:

$$q_r^{"} = -k\frac{\partial T}{\partial x}$$

φ component flux is:

$$q_\varphi^{"} = -\frac{k}{r}\frac{\partial T}{\partial \varphi}$$

and z component flux is:

$$q_z^{"} = -k\frac{\partial T}{\partial z}$$

These are the things where now if we look at the generic expression for it, the generic expression for the cylindrical coordinate is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(k\frac{\partial T}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}$$

So, this form is for the cylindrical coordinate, where each and every term is shown here and it follows the similar analysis which will not go through at this moment.

And, similarly when we talk about the spherical coordinate for the spherical coordinate it takes a form:

$$q^{"} = -k\nabla T$$

$$= -k\left(i\frac{\partial T}{\partial r} + j\frac{1}{r}\frac{\partial T}{\partial \theta} + k\frac{1}{rsin\theta}\frac{\partial T}{\partial \varphi}\right)$$

So, this is for the spherical coordinate (r, θ, φ) direction. It takes a form which is something like:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2sin^2\theta}\frac{\partial}{\partial \varphi}\left(k\frac{\partial T}{\partial \varphi}\right) + \frac{1}{r^2sin\theta}\frac{\partial}{\partial \theta}\left(ksin\theta\frac{\partial T}{\partial \varphi}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}$$

This form is for the spherical coordinate which again has all the components in all the directions and its expressions. Again, these are the generic form in the cylindrical and the spherical coordinates like that we have seen for the Cartesian coordinates.

So, this is for today. I will stop here and the next class, I will come back with a few example which will be helpful for you to understand the application of this forms.

With this, thank you for your attention.