Chemical Engineering Fluid Dynamics and Heat Transfer

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Lecture - 29
Turbulence 07

So, welcome back. In the previous lecture we have discussed the concept Prandtl mixing length and One of the utilities of Prandtl mixing length is that it correlates the fluctuation component of the velocity gradients to the gradient of the mean velocities. This is actually to reduce the number of variables to resolve the conservation equations.

Most popular expression of Prandtl mixing length, for the Flow over a flat plate:

Prandtl mixing length, $L = Ky$

$K \rightarrow Van\ karmann\ constant: y \rightarrow distance\ from\ the\ wall$

Note: Typically for many cases it has been experimentally obtained that the value of K is approximately 0.4.

$\tau_{xy_T} = -\rho\, \overline{u'v'}$

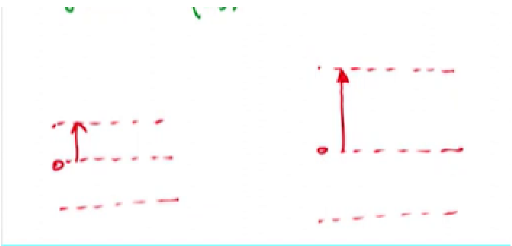
For turbulent stress, $\tau_{xy_T} = \rho L^2 (\frac{\partial \bar{u}}{\partial y})^2 = \rho K^2 y^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2$

And we have

$$L = \left(\frac{\left(\frac{\partial \bar{u}}{\partial y}\right)}{\epsilon}\right)^{-\frac{1}{2}}$$

Form this we have expression for eddy diffusivity,

$$\epsilon = L^2 \left(\frac{\partial \bar{u}}{\partial y}\right)$$



Case 1 Case 2

Because of fluctuation the fluid particle will get shifted from one level to higher level. So, let us say that from j th level to the j+1 st level by creating a velocity gradient between these two levels. This velocity gradient that gets created is a function of the fluctuation component itself. In the first case is an example for lower intensity turbulence and the second case is for higher intensity turbulence. So, that means, that the strength of the fluctuating component of velocity in the first case is lower and in the second case it is higher. Therefore, in the first case the fluid particle will get shifted by certain amount because of the fluctuating component of velocity. But in the second case the strength of the fluctuation is higher, and the fluid is going to be shifted by a higher amount.

If the fluctuation is higher, particle get shifted by a higher distance and that means, how much this particle is going to be shifted is qualitatively captured by the Prandtl mixing length. So, this is one of the utilities of correlating Prandtl mixing length with the intensity of the turbulence.

<u>Turbulent wall friction:</u>

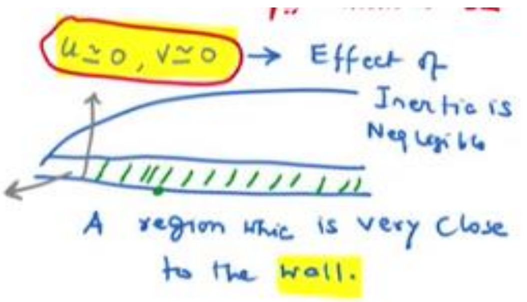
x component Navier-Stokes equation within a boundary layer

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \mu\frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\partial}{\partial y}(\tau_{xy})$$

for the turbulent flow we can write the x component Navier-Stokes equation within a boundary layer by incorporating Reynolds decomposition in the above equation and by taking time average. Then we will get

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \mu\frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\partial}{\partial y}(\tau_{xy_E})$$

Here the turbulent stress term comprises of turbulent stress as well as viscous stress. We are considering the region that is very close to the wall, within the region because of the no slip condition,$u \approx 0, and\ v \approx 0$ due to the semi-impermeable wall.



$\frac{1}{\rho}\frac{\partial}{\partial y}(\tau_{xy_E}) = 0$

$$\rightarrow \frac{1}{\rho}\frac{d}{dy}(\tau_{xy_E}) = 0$$

Therefore we can say that

$$(\tau_{xy_E}) = constant.$$

τ_{xy_E} does not vary with y.within this region, $\tau_{xy_E} = \tau_w = constant$

And we know that, Effective turbulent stress,

$$\tau_{xy_E} = \rho(\gamma + \epsilon)\,\frac{\partial \bar{u}}{\partial y} = (\mu + \mu_T)\,\frac{\partial \bar{u}}{\partial y}$$

$$\frac{\tau_{xy_E}}{\rho} = (\gamma + \epsilon)\frac{\partial \bar{u}}{\partial y}$$

$$\frac{\tau_w}{\rho} = (\gamma + \epsilon)\frac{\partial \bar{u}}{\partial y}$$

This region is called constant apparent / effective shear stress region.

When we check the unit of $\tau_w = \frac{kgf}{m^2}, \rho = \frac{kgm}{m^3}$ then $\frac{\tau_w}{\rho} = \frac{m^2}{s^2}$

In the case of Force, $P \propto m.a \Rightarrow P = k.m.a \Rightarrow P = m.a$, k = 1

If force is applied on an object of unit mass and it produces unit acceleration.

$P = \frac{1}{g_c}m.a \rightarrow [g_c] = \frac{kgm.m}{kgf.s^2}$

Unit of $\frac{\tau_w}{\rho} = (unit\ of\ velocity)^2$

$U_* or\ U_\tau \rightarrow Friction\ velocity\ = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$

This friction velocity is used for the following nondimensionalization.

Wall velocities:

$u^+ = \frac{\bar{u}}{u_*}, v^+ = \frac{\bar{v}}{v_*},$

Wall coordinates:

$x^+ = \frac{x u_*}{\gamma}, y^+ = \frac{y u_*}{\gamma}$

$$\frac{\tau_w}{\rho} = (\gamma + \epsilon)\frac{\partial \bar{u}}{\partial y} : \frac{\partial u^+}{\partial y^+} = \left(\frac{\partial u^+}{\partial y}\right)\left(\frac{\partial y}{\partial y^+}\right)$$

$y^+ = \frac{y u_*}{\gamma} : y = \frac{y^+ \gamma}{u_*}$: from this $\left(\frac{\partial y}{\partial y^+}\right) = \frac{\gamma}{u_*}$

$$\frac{\partial u^+}{\partial y^+} = \left(\frac{\partial u^+}{\partial y}\right)\left(\frac{\partial y}{\partial y^+}\right) = \left(\frac{\partial u^+}{\partial y}\right)\frac{\gamma}{u_*}$$

$$= \frac{\partial}{\partial y}\left(\frac{\overline{u}}{u_*}\right)\frac{\gamma}{u_*} = \frac{\partial \bar{u}}{\partial y} . \frac{\gamma}{(u_*)^2}$$

Therefore,

$$\frac{\partial \bar{u}}{\partial y} = \frac{(u_*)^2}{\gamma}\frac{\partial u^+}{\partial y^+}$$

$$\frac{\tau_w}{\rho} = (\gamma + \epsilon)\frac{\partial \bar{u}}{\partial y}$$

$$(\gamma + \epsilon)\frac{(u_*)^2}{\gamma}\frac{\partial u^+}{\partial y^+} = \frac{\tau_w}{\rho}$$

From this relation, $u_* or\ u_\tau \rightarrow Friction\ velocity\ = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$

we can write, $u_* or\ u_\tau = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$:therefore, $\frac{\tau_w}{\rho} = (u_*)^2$

if we substitute this in the above equation we will get

$$(\gamma + \epsilon)\frac{(u_*)^2}{\gamma}\frac{\partial u^+}{\partial y^+} = (u_*)^2$$

$$(\gamma + \epsilon)\frac{1}{\gamma}\frac{\partial u^+}{\partial y^+} = 1$$

$$\left(1 + \frac{\epsilon}{\gamma}\right)\frac{\partial u^+}{\partial y^+} = 1$$

This is the simplified momentum balance equation in wall coordinate system within a region that is very close to the wall where the inertial effects are completely neglected.

Thank you very much.