Chemical Engineering Fluid Dynamics and Heat Transfer
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Lecture - 26
Turbulence 04

So, welcome back. We are going to plug the Reynolds decomposition in the momentum balance equation.

Consider the momentum balance equation,

x component Navier Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Consider as steady state, the term $\frac{\partial u}{\partial t}$ will be zero.

We take the conservative form of the Navier-Stokes equation for the x component, by adding the term $u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$ to the L.H.S.

Then we will get,

$$\rho\left(\frac{\partial u^2}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

As velocity fluctuates pressure also fluctuates. We will introduce the Reynolds decomposition into the terms of the above equation. Therefore, all the terms that are contributing to momentum transfer or transport should be present, that is the reason we take the conservative form.

$$\rho\left(\frac{\partial(\bar{u}+u')^2}{\partial x}+\frac{\partial\left((\bar{u}+u')\left((\bar{v}+v')\right)\right)}{\partial y}+\frac{\partial\left((\bar{u}+u')(\bar{w}+w')\right)}{\partial z}\right)$$
$$=-\frac{\partial(\bar{p}+p')}{\partial x}+\mu\left(\frac{\partial^2(\bar{u}+u')}{\partial x^2}+\frac{\partial^2(\bar{u}+u')}{\partial y^2}+\frac{\partial^2(\bar{u}+u')}{\partial z^2}\right)$$

We take time average of the above equation,

$$\overline{\rho\left(\frac{\partial(\bar{\mathrm{u}}+\mathrm{u}')^2}{\partial x}+\frac{\partial\left((\bar{\mathrm{u}}+\mathrm{u}')\left((\bar{\mathrm{v}}+\mathrm{v}')\right)\right)}{\partial y}+\frac{\partial\left((\bar{\mathrm{u}}+\mathrm{u}')(\bar{\mathrm{w}}+\mathrm{w}')\right)}{\partial z}\right)}$$

$$=-\overline{\frac{\partial(\bar{\mathrm{p}}+\mathrm{p}')}{\partial x}}+\overline{\mu\left(\frac{\partial^2(\bar{\mathrm{u}}+\mathrm{u}')}{\partial x^2}+\frac{\partial^2(\bar{\mathrm{u}}+\mathrm{u}')}{\partial y^2}+\frac{\partial^2(\bar{\mathrm{u}}+\mathrm{u}')}{\partial z^2}\right)}$$

R.H.S of the equation,

$$-\frac{-\partial\overline{(\bar{\mathrm{p}}+\mathrm{p}')}}{\partial x}=-\frac{-\partial\left(\bar{\bar{p}}+\overline{p'}\right)}{\partial x}=-\frac{\partial\bar{p}}{\partial x}$$

$$\overline{\mu\frac{\partial^2(\bar{\mathrm{u}}+\mathrm{u}')}{\partial x^2}}=\mu\frac{\partial^2}{\partial x^2}(\bar{\bar{u}})+\mu\frac{\partial^2}{\partial x^2}(\overline{u'})$$

Since the term $\mu\frac{\partial^2}{\partial x^2}(\overline{u'})$ becomes zero, therefore $\overline{\mu\frac{\partial^2(\bar{\mathrm{u}}+\mathrm{u}')}{\partial x^2}}=\mu\frac{\partial^2\bar{u}}{\partial x^2}$

Similarly, the RHS of the above equation becomes,

$$-\frac{\partial(\bar{\mathrm{p}})}{\partial x}+\mu\left(\frac{\partial^2(\bar{\mathrm{u}})}{\partial x^2}+\frac{\partial^2(\bar{\mathrm{u}})}{\partial y^2}+\frac{\partial^2(\bar{\mathrm{u}})}{\partial z^2}\right)$$

Time averaging of the L.H.S terms,

First, we will look into the second term of the L.H.S,

$$\overline{\frac{\partial}{\partial y}\left((\bar{\mathrm{u}}+\mathrm{u}')(\bar{\mathrm{v}}+\mathrm{v}')\right)}=\frac{\partial}{\partial y}\left(\overline{\bar{u}.\bar{v}+\bar{u}.v'+u'.\bar{v}+u'v'}\right)$$

$$=\frac{\partial}{\partial y}\left(\overline{\bar{u}.\bar{v}}+\overline{\bar{u}.v'}+\overline{u'.\bar{v}}+\overline{u'v'}\right)$$

$$=\frac{\partial}{\partial y}\left(\overline{uv}+\overline{u'v'}\right)$$

First term of the L.H.S,

$$\frac{\partial}{\partial x}\overline{(\bar{\mathrm{u}}+\mathrm{u}')(\bar{\mathrm{u}}+\mathrm{u}')}=\frac{\partial}{\partial x}\left(\bar{u}^2+\overline{u'^2}\right)$$

Third term of the L.H.S,

$$\frac{\partial}{\partial z}\overline{\left((\bar{u}+u')(\bar{w}+w')\right)}=\frac{\partial}{\partial z}\left(\overline{uw}+\overline{u'w'}\right)$$

If we incorporate the term in the x component balance equation,

$$\rho\left(\frac{\partial}{\partial x}\left(\bar{u}^2+\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\overline{uv}+\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\overline{uw}+\overline{u'w'}\right)\right)=-\frac{\partial(\bar{p})}{\partial x}+\mu\left(\frac{\partial^2(\bar{u})}{\partial x^2}+\frac{\partial^2(\bar{u})}{\partial y^2}+\frac{\partial^2(\bar{u})}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial}{\partial x}\bar{u}^2+\frac{\partial(\overline{uv})}{\partial y}+\frac{\partial(\overline{uw})}{\partial z}\right)+\rho\left(\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\left(\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right)$$
$$=-\frac{\partial(\bar{p})}{\partial x}+\mu\left(\frac{\partial^2(\bar{u})}{\partial x^2}+\frac{\partial^2(\bar{u})}{\partial y^2}+\frac{\partial^2(\bar{u})}{\partial z^2}\right)$$

$$\underbrace{\rho\left(\frac{\partial}{\partial x}\bar{u}^2+\frac{\partial(\overline{uv})}{\partial y}+\frac{\partial(\overline{uw})}{\partial z}\right)}_{\text{Inertial terms}}+\rho\left(\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\left(\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right)=\underbrace{-\frac{\partial(\bar{p})}{\partial x}}_{\text{Pressure}}+\mu\underbrace{\left(\frac{\partial^2(\bar{u})}{\partial x^2}+\frac{\partial^2(\bar{u})}{\partial y^2}+\frac{\partial^2(\bar{u})}{\partial z^2}\right)}_{\text{Skin friction term}}$$

$$\rho\left(\frac{\partial}{\partial x}\bar{u}^2+\frac{\partial(\overline{uv})}{\partial y}+\frac{\partial(\overline{uw})}{\partial z}\right)=-\frac{\partial(\bar{p})}{\partial x}+\mu\left(\frac{\partial^2(\bar{u})}{\partial x^2}+\frac{\partial^2(\bar{u})}{\partial y^2}+\frac{\partial^2(\bar{u})}{\partial z^2}\right)-\rho\left(\frac{\partial}{\partial x}\overline{u'^2}+\frac{\partial}{\partial y}\left(\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right)$$

If we write regular functional form:

$$\underbrace{\rho\left(\frac{\partial u^2}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}\right)}_{\text{Inertial/Additive term}}=\underbrace{-\frac{\partial p}{\partial x}}_{\text{Pressure}}+\mu\underbrace{\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)}_{\text{Skin friction term}}$$

Thank you very much.