Chemical Engineering Fluid Dynamics and Heat Transfer
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Lecture - 21
Boundary Layer Analysis 6: Momentum Integral Method 02

So, welcome back. In today's class we will start discussing about the Momentum Integral Method. Here are the expressions for the boundary layer displacement thickness and the momentum thickness.

Boundary layer displacement thickness, $\delta^* = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy$

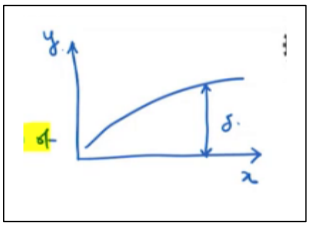
Boundary layer momentum thickness, $\theta = \delta^{**} = \int_0^\infty \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy$

$$H = \frac{\delta^*}{\theta} = Shape\ factor\ of\ the\ boundary\ layer\ or\ shape\ factor$$

<u>Momentum integral method:</u>

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \gamma\left(\frac{\partial^2 u}{\partial y^2}\right)$$

Integrate the above equation over the thickness of boundary layer



$$\int_0^\delta \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) dy = \int_0^\delta \gamma\left(\frac{\partial^2 u}{\partial y^2}\right) dy$$

$$\int_0^\delta u\frac{\partial u}{\partial x} dy + \int_0^\delta v\frac{\partial u}{\partial y} dy = \int_0^\delta \gamma\left(\frac{\partial^2 u}{\partial y^2}\right) dy$$

First, we look into the second term of the L.H.S of the equation,

We know: $\frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$

Therefore, the second term of the L.H.S becomes,

$$\int_0^\delta v\frac{\partial u}{\partial y} dy = \left[\int_0^\delta \left(\frac{\partial}{\partial y}(uv) - u\frac{\partial v}{\partial y}\right)\right] dy$$

$$= \int_0^{\delta} \left(\frac{\partial}{\partial y}(uv)dy\right) - \int_0^{\delta} \left(u\frac{\partial v}{\partial y}dy\right)$$

And we know that from continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

Then we can write,

$$\int_0^{\delta} v\frac{\partial u}{\partial y}dy = [(uv)]|_0^{\delta} + \int_0^{\delta} u\left(\frac{\partial u}{\partial x}\right)dy$$

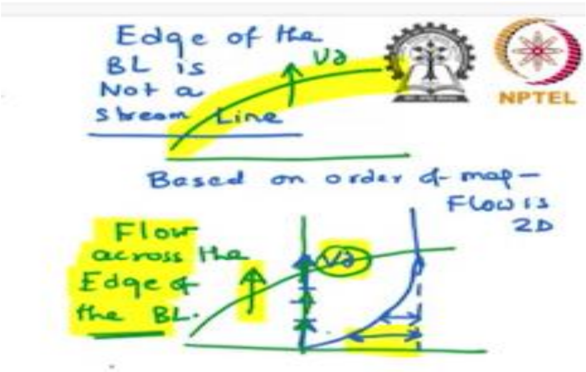
$$= (uv)|_{\delta} - (uv)|_0 + \int_0^{\delta} u\left(\frac{\partial u}{\partial x}\right)dy$$

$$= u_{\infty}v_{\delta} - u_0 v_0 + \int_0^{\delta} u\left(\frac{\partial u}{\partial x}\right)dy$$

we know $u_0 = v_0 = 0$,Therefore second term on L.H.S,

$$\int_0^{\delta} v\frac{\partial u}{\partial y}dy = u_{\infty}v_{\delta} + \int_0^{\delta} u\left(\frac{\partial u}{\partial x}\right)dy$$

v_{δ} is the total flow due to the y component velocity. We know that based on order of magnitude analysis that flow is 2D inside the boundary layer, that means v is nonzero. And, in the context of discussing the boundary layer displacement thickness we understood that because of the boundary layer separation at different x certain amount of fluid is not flowing. It was supposed to flow being potential or had there been no boundary layer formation, but that is not flowing because of the boundary layer separation. So, these fluids that are b not flowing through this area has been retarded out and they actually lead to the y component velocity. And y component velocity at every level has two components, one is the flow that it is inheriting from the lower level and the second component is the additional amount of flow that is getting augmented because of the retardation at that particular level. So, basically the summation of the cumulative sum of all these mass that is not flowing or the fluid that eventually drains out of the boundary layer and joins the bulk flow through this v_{δ} . all the fluid that are not flowing at different levels actually flow out of the boundary layer through this v_{δ}. So, whatever is the retardation due to the formation of the boundary layer that flow, whatever is the fluid that is unable to flow through that particular level because of the boundary layer formation that eventually joins that y component velocity and eventually drains out through the edge of the boundary layer that is v_δ.so we can write $v_\delta = \int_0^\delta \left(\frac{\partial v}{\partial y}\right) dy$.



There is flow across the edge of the boundary layer and that means, since there is flow across the edge of the boundary, the edge of the boundary layer cannot be a streamline.

Then the L.H.S of the equation can be written as,

$$\int_0^\delta v\frac{\partial u}{\partial y}dy = u_\infty \int_0^\delta \left(\frac{\partial v}{\partial y}\right) dy + \int_0^\delta u\left(\frac{\partial u}{\partial x}\right) dy$$

$$= -u_\infty \int_0^\delta \left(\frac{\partial u}{\partial x}\right) dy + \int_0^\delta u\left(\frac{\partial u}{\partial x}\right) dy$$

The L.H.S of the integrated equation becomes,

$$\int_0^\delta u\frac{\partial u}{\partial x}dy + \int_0^\delta v\frac{\partial u}{\partial y}dy = \int_0^\delta u\frac{\partial u}{\partial x}dy - u_\infty \int_0^\delta \left(\frac{\partial u}{\partial x}\right) dy + \int_0^\delta u\left(\frac{\partial u}{\partial x}\right) dy$$

$$= 2\int_0^\delta u\left(\frac{\partial u}{\partial x}\right) dy - u_\infty \int_0^\delta \left(\frac{\partial u}{\partial x}\right) dy$$

$$= \int_0^\delta \frac{d}{dx} u(u - u_\infty) dy$$

$$= -{u_\infty}^2 \int_0^\delta \frac{d}{dx}\left(\frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)\right) dy$$

We know the relation for boundary layer momentum thickness, $\theta = \int_0^\infty \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy$,

$$\int_0^\delta u\frac{\partial u}{\partial x}dy + \int_0^\delta v\frac{\partial u}{\partial y}dy = -{u_\infty}^2 \frac{d\theta}{dx}$$

We will look into the R.H.S of the equation,

$$\int_0^\delta \gamma\left(\frac{\partial^2 u}{\partial y^2}\right)dy = \left[\gamma.\frac{\partial u}{\partial y}\right]_0^\delta = \gamma.\frac{\partial u}{\partial y}|_\delta - \gamma.\frac{\partial u}{\partial y}|_0 = \frac{-\tau_w}{\rho}$$

Note: at the edge $u = u_\infty, \frac{\partial u}{\partial y}|_{y=\delta} = 0$:And we know that $\tau_w = \mu\frac{\partial u}{\partial y}$

By combining the L.H.S and R.H.S,

$$-u_\infty{}^2\frac{d\theta}{dx} = \frac{-\tau_w}{\rho}$$

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho u_\infty{}^2}$$

We have to evaluate θ ,for that we need to assume a velocity profile. In order to perform the momentum integral method, we have to assume a velocity profile. This is the limitation of the momentum integral method, though the method is often referred to as an approximate method.

Karman-Pahlausen approximate method for solving the momentum integral equation

We assume a polynomial velocity profile:

$$\frac{u}{u_\infty} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3$$

And we know that $\eta = \frac{y}{\delta}$

To evaluate the constant, we will use the boundary conditions:

1.At $y = 0, u = 0(due\ to\ no\ slip\ condition)$ →At $\eta = 0, \frac{u}{u_\infty} = 0$

2.At $y = 0, \frac{\partial^2 u}{\partial y^2} = 0(constant\ wall\ shear\ stress\ at\ the\ wall$ →At $\eta = 0, \frac{\partial^2}{\partial \eta^2}\left(\frac{u}{u_\infty}\right)0$

3.At $y = \delta, u = u_\infty$ →At $\eta = 1, \left(\frac{u}{u_\infty}\right) = 1$

4. At $y = \delta, \frac{\partial u}{\partial y} = 0(\text{there is no variation of velocity along y } at \text{ the edge. Therefore,}$

at $\eta = 1, \frac{\partial}{\partial \eta}\left(\frac{u}{u_\infty}\right) = 0$

By using the above boundary conditions, we will get

$$a_0 = 0, a_2 = 0$$

$$a_1 + 3a_3 = 0$$

$$a_1 + a_3 = 1$$

By solving them, we will get the velocity profile as

$$\frac{u}{u_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$

This is the velocity profile.

Thank you.