Chemical Engineering Fluid Dynamics and Heat Transfer
Prof. Rabibrata Mukherjee
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 19
Boundary Layer Analysis 4: Blasius Solution 2

So, welcome back to our 19th lecture and we were discussing the Blasius Solution or essentially, how to solve the Boundary Layer Equation.

From Falkner skan transformation, we got that

$$\psi = \sqrt{\gamma x u_\infty}.\bar{f}(\eta)$$

$$\frac{\partial \psi}{\partial \eta} = \sqrt{\gamma x u_\infty}.\frac{df}{d\eta}$$

$$\eta = \sqrt{\frac{u_\infty}{\gamma x}}.y$$

$$\frac{\partial \eta}{\partial x} = \left(-\frac{1}{2}\sqrt{\frac{u_\infty}{\gamma}}\ x^{-\frac{3}{2}}\right)y = \left(-\frac{1}{2}\sqrt{\frac{u_\infty}{\gamma.x}}.\frac{y}{x}\right) = \left(-\frac{1}{2}\frac{\eta}{x}\right)$$

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{u_\infty}{\gamma x}}$$

Based on stream function, we can write that

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}.\frac{\partial \eta}{\partial y} = \sqrt{\gamma x u_\infty}.\frac{df}{d\eta}.\sqrt{\frac{u_\infty}{\gamma x}}$$

$$u = u_\infty \frac{df}{d\eta}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(\sqrt{\gamma x u_\infty}.\bar{f}(\eta)\right)$$

$$= -\left(\sqrt{\gamma x u_\infty}.\frac{\partial f}{\partial x} + \frac{1}{2}\sqrt{\frac{\gamma u_\infty}{x}}.f\right)$$

$$= -\left(\sqrt{\gamma x u_\infty}.\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{1}{2}\sqrt{\frac{\gamma u_\infty}{x}}.f\right)$$

$$= -\left(\sqrt{\gamma x u_\infty}.\frac{df}{d\eta}.\left(-\frac{1}{2}\frac{\eta}{x}\right) + \frac{1}{2}\sqrt{\frac{\gamma u_\infty}{x}}.f\right)$$

$$v = \frac{1}{2}\sqrt{\frac{\gamma u_\infty}{x}}\left(\eta\frac{df}{d\eta} - f\right)$$

$$u = u_\infty\frac{df}{d\eta}$$

If we differentiate u w.r.t to x,

$$\frac{\partial u}{\partial x} = u_\infty\frac{\partial}{\partial x}\left(\frac{df}{d\eta}\right)$$

$$= u_\infty\frac{d}{d\eta}\left(\frac{df}{d\eta}\right).\frac{\partial \eta}{\partial x}$$

$$= u_\infty\frac{d^2 f}{d\eta^2}\left(-\frac{1}{2}\frac{\eta}{x}\right)$$

$$\frac{\partial u}{\partial x} = -\frac{u_\infty}{2x}.\eta\frac{d^2 f}{d\eta^2}$$

If we differentiate u w.r.t to y,

$$\frac{\partial u}{\partial y} = u_\infty\frac{\partial}{\partial y}\left(\frac{df}{d\eta}\right)$$

$$= u_\infty\frac{d}{d\eta}\left(\frac{df}{d\eta}\right).\frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial y} = u_\infty\sqrt{\frac{u_\infty}{\gamma x}}\frac{d^2 f}{d\eta^2}$$

Note: $f\ and\ f(\eta)\ are\ same.$

If we differentiate $\frac{\partial u}{\partial y}$ w.r.t to y again,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{d}{d\eta}\left(\frac{\partial u}{\partial y}\right).\left(\frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{d}{d\eta}.\sqrt{\frac{u_\infty}{\gamma x}}\frac{d^2 f}{d\eta^2}.\sqrt{\frac{u_\infty}{\gamma x}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{{u_\infty}^2}{\gamma . x}\frac{d^3 f}{d\eta^3}$$

$$v = \frac{1}{2}\sqrt{\frac{\gamma u_\infty}{x}}\left(\eta\frac{df}{d\eta} - f\right)$$

From the boundary layer equation,

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \gamma\left(\frac{\partial^2 u}{\partial y^2}\right)$$

If we substitute for u and v in term of f and η,we get the L.H.S term of the above equation:

$$\left(u_\infty\frac{df}{d\eta}\right)\left(-\frac{u_\infty}{2x}.\eta\frac{d^2 f}{d\eta^2}\right) + \frac{1}{2}\sqrt{\frac{\gamma u_\infty}{x}}\left(\eta\frac{df}{d\eta} - f\right).\left(u_\infty\sqrt{\frac{u_\infty}{\gamma x}}\frac{d^2 f}{d\eta^2}\right)$$

$$= -\frac{1}{2}\frac{{u_\infty}^2}{x}.\eta\frac{df}{d\eta}.\frac{d^2 f}{d\eta^2} + \frac{1}{2}\frac{{u_\infty}^2}{x}\eta\frac{df}{d\eta}.\frac{d^2 f}{d\eta^2} - \frac{1}{2}\frac{{u_\infty}^2}{x}f\frac{d^2 f}{d\eta^2}$$

$$= -\frac{1}{2}\frac{{u_\infty}^2}{x}f\frac{d^2 f}{d\eta^2}$$

R.H.S term of the above equation:

$$\gamma\left(\frac{\partial^2 u}{\partial y^2}\right) = \gamma\frac{{u_\infty}^2}{\gamma . x}\frac{d^3 f}{d\eta^3} = \frac{{u_\infty}^2}{x}\frac{d^3 f}{d\eta^3}$$

If we substitute these terms in the boundary layer equation, finally we get

$$-\frac{1}{2}\frac{{u_\infty}^2}{x}f\frac{d^2 f}{d\eta^2} = \frac{{u_\infty}^2}{x}\frac{d^3 f}{d\eta^3}$$

$$2\frac{d^3 f}{d\eta^3} + f\,\frac{d^2 f}{d\eta^2} = 0$$

Now, this is a third order ordinary differential equation. Since, it is third order equation, we need three boundary conditions, and the boundary conditions are:

1.At $\eta = 0, f = 0: \eta = \left(\frac{y}{\delta}\right), if \eta = 0, then\ y = 0$ → on the surface of the plate

2.At $\eta = 0: \frac{df}{d\eta} = 0$

3.At $\eta = \infty: \frac{df}{d\eta} = 1$

$$2\frac{d^3f}{d\eta^3} + f\ \frac{d^2f}{d\eta^2} = 0$$

This can be split up to 3 coupled first order ODE. And the solution of the above equation by numerical approach, we will get:

$$f(\eta) = \frac{0.332}{2!}\eta^2 - \frac{1}{2}.\frac{(0.332)^2}{5!}\eta^5 + \frac{11}{4}.\frac{(0.332)^3}{8!}\eta^8 \ ....$$

One of the utilities of the above equation is to find out the wall shear stress,

$$\tau_w = 0.332\mu u_\infty \sqrt{\frac{u_\infty}{\gamma x}}$$

We will discuss bout this in detail in the next lecture.

Thank you very much.