Chemical Engineering Fluid Dynamics and Heat Transfer
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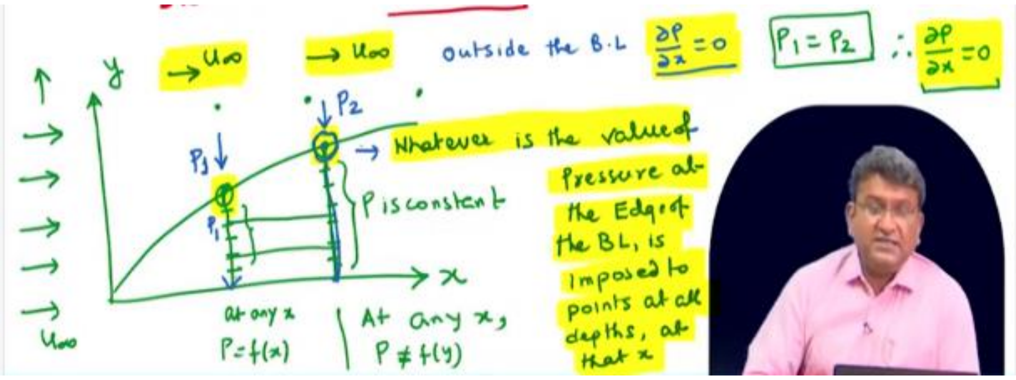
Lecture - 17
Boundary Layer Analysis 3

So, welcome back. From order of magnitude analysis for the continuity equation, we get to know that order the y component velocity is very low or much lower as compared to the order of x component velocity. And also, the order of the y component velocity is non-zero. That means is that the flow within a boundary layer is two dimensional. Then we looked into the order of magnitude analysis of the momentum balance equations. And we simplified equations for the x component and the y component momentum balance equation for boundary layer. Both the equations have generically three terms one is the inertia term, one is the pressure term, and one is the skin friction or the stress term. If the pressure term is active or alive within the equation its order will be similar to the order of the skin friction term and inertial term.

From the previous lecture, we got to know pressure is a function of x only.



$$P = f(x)$$

So, at any particular x, P does not vary with y. So, the value of P is a function of x only. At any x, value of P is constant. The value of pressure outside the boundary layer or at the edge of the boundary layer is actually the same. So, whatever the value of pressure at the edge of the boundary layer is imposed to all points of the depths at that x.

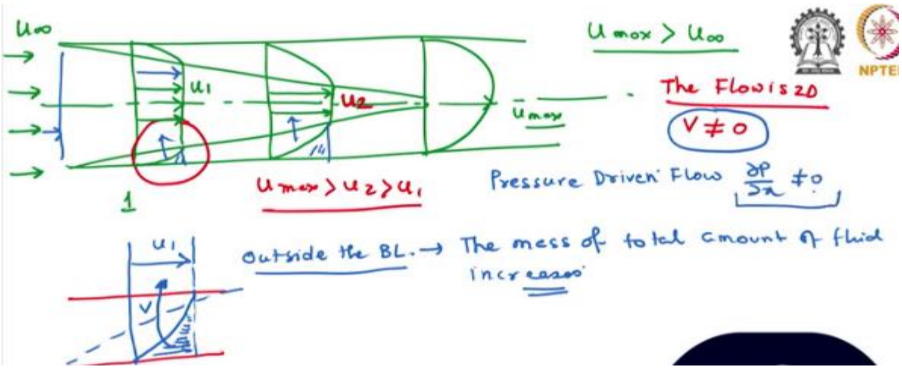
So, if the outside pressure at the edge of the boundary layer is P_1 at x = x_1, therefore at all the points along the depth of the boundary layer the pressure is P_1 at x = x_1. Similarly, if the outside pressure here is P_2 at all the points along the depth of the boundary layer the pressure is P_2 at the point x = x_2.

The one assumption, we are considering that velocity will be u_∞ outside the boundary layer and there is no pressure drop along the so, if there is no pressure gradient outside the boundary layer that would mean that P_1 is equal to P_2 or in other words the pressure at every point of the edge of the boundary layer is same irrespective of x.

For this particular system, within the boundary layer $P_1 = P_2$: $\frac{\partial P}{\partial x} = 0$

Because there is no variation in pressure outside this is a special case and it is limited to this case only.

If you consider the flow through the pipe/tube, assumption $\frac{\partial P}{\partial x} = 0$ is not valid in this case.The flow velocity of fluid entering to the pipe is u_∞ and after crossing the transition length, the velocity becomes u_{max} and this u_{max} is actually higher than u_∞. Because as the boundary layer starts to form, as the fluid enters the pipe and within the boundary layer the flow becomes 2D, which means v is non-zero. As the boundary layer starts to grow, the velocity of the fluid entering the pipe changes. So, some the amount of fluid would have gone or passed through the boundary would not be able to flow. So, this fluid is transported back to outside of the boundary layer by the y component velocity (the non-zero v). So, outside the boundary layer the mass of total amount of fluid increases. Therefore, from point to point the amount of fluid that is pushed out has to flow and the zone where the plug flow existing becomes narrower in diameter therefore, the velocity keeps on increasing. This is a pressure driven flow and $\frac{\partial P}{\partial x} \neq 0$



Final set of Prandtl boundary layer equation

1.$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (continuity equation)

2.$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \gamma\left(\frac{\partial^2 u}{\partial y^2}\right)$ (x component equation simplified)

3.$\frac{\partial P}{\partial y} = 0$(obtained from y component balance)

One of the terms in the y component balance becomes zero, therefore the order of all other terms, inertia term and skin friction/stress term becomes zero. If we look into this equation,

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \gamma\left(\frac{\partial^2 u}{\partial y^2}\right)$$

the unknowns are only two: u and v and we have two equations, so these equations can be solved analytically.

x component balance equation will be,

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \gamma\left(\frac{\partial^2 u}{\partial y^2}\right)$$

The pressure term becomes zero.

Inertia terms→ $\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) \rightarrow \frac{{u_\infty}^2}{L}$

Skin friction or stress term→ $\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial y^2}\right) \rightarrow \gamma\frac{u_\infty}{\delta^2}$

Order of inertial term = Order of skin friction/stress term

$$\frac{{u_\infty}^2}{L} \cong \gamma \frac{u_\infty}{\delta^2}$$

$$\frac{\delta^2}{L} \cong \frac{\gamma}{u_\infty}$$

$$\frac{\delta^2}{L^2} \cong \frac{\gamma}{u_\infty . L} \cong \frac{\mu}{\rho u_\infty . L}$$

And we know that, Reynold no for pipe flow,$Re = \frac{Dv\rho}{\mu}$

Therefore, above equation can be written as $Re,_L = \frac{\rho u_\infty L}{\mu}$

$$Re,_x = \frac{\rho u_\infty x}{\mu}$$

So, this is the relation for Reynolds number across the entire length for the flow and as the flow moves from x_1 to x_2,the value of Reynolds number changes.

$$\frac{\delta^2}{L^2} \cong \frac{\gamma}{u_\infty . L} \cong \frac{\mu}{\rho u_\infty . L}$$

$$\left(\frac{\delta}{L}\right)^2 = \frac{1}{Re,_L}$$

$$\left(\frac{\delta}{L}\right) = (Re,_L)^{-\frac{1}{2}}$$

$$\delta = L.(Re,_L)^{-\frac{1}{2}}$$

For any point x,

$$\delta = x.(Re,_x)^{-\frac{1}{2}}$$

And we know that, $\delta \ll L$

$$(Re_{,L})^{-\frac{1}{2}} \ll 1$$

Or

$$(Re_{,L})^{\frac{1}{2}} \gg 1$$

$Re_{,L} \gg 1$,This analysis is valid for high Reynolds number flows.

So, unlike the classical Reynolds number, which is the ratio between the inertial force to the viscous force, in the context of a boundary layer the $Re_{,L}$ is a geometric parameter. It is a geometric parameter that actually gives the ratio of the length of the plate over which the boundary layer separation takes place to the thickness of the boundary layer.

$$Re_{,L} = \left(\frac{L}{\delta}\right)^2$$

For same plate length, if $Re_{,L1} > Re_{,L2} \rightarrow \delta_1 < \delta_2$, the thickness of the boundary layer is thinner in the first case than the second one.

$$\left(\frac{\delta}{L}\right) = (Re_{,L})^{-\frac{1}{2}}$$

$$\delta = L.\frac{\gamma^{\frac{1}{2}}}{L^{\frac{1}{2}}.u_{\infty}^{\frac{1}{2}}}$$

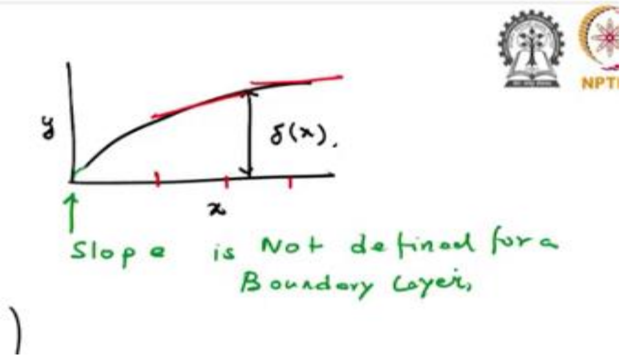
So, for any x,

$$\delta = x^{\frac{1}{2}}\frac{\gamma^{\frac{1}{2}}}{u_{\infty}^{\frac{1}{2}}}$$

$$\frac{d\delta}{dx} = \frac{1}{2}.\left(\frac{1}{x^{\frac{1}{2}}}\right)\left(\frac{\gamma^{\frac{1}{2}}}{u_{\infty}^{\frac{1}{2}}}\right)$$

This is the slope of the curve that represents the edge of the boundary layer.At x=0, means at the leading edge of the plate, the slope of the boundary layer is not defined.

$$\frac{d\delta}{dx}|_{x=0} = undefined.$$



Based on order of magnitude analysis, we understand that within the boundary layer, the flow nature is 2D flow and $\frac{\partial P}{\partial x}$ is equal to 0. Next class we will discuss how we will solve the boundary layer equations analytically.

Thank you very much.