Chemical Engineering Fluid Dynamics and Heat Transfer
Prof. Rabibrata Mukherjee
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 16
Boundary Layer Analysis 2

So, welcome back to our discussion on Boundary Layers.

when a 1D flow passes over a solid plate and there is boundary layer separation. And because of the boundary layer separation, within the boundary layer the flow becomes 2D. Though, the y component velocity is found to be much smaller than x component velocity based on an order of magnitude analysis. While doing order of magnitude analysis of any equation you cannot neglect any term which has it is genesis in mass balance.

However, for other type of equation that is in our context if you are looking at equation it is like momentum balance you can neglect similar terms with respect to other terms if they are found to be much smaller. And based on this introduction let us see how we do the scaling analysis or order of magnitude analysis for the momentum balance equations over a boundary layer. So, basically, we have already looked into the order of magnitude analysis for the continuity equation.

Now, let us look into the momentum balance equations for 2D Newtonian fluid,

x component balance equation:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

y component balance equation:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

We define the non-gravitational pressure, $P = p + \rho g y$

$$\frac{\partial P}{\partial y}=\frac{\partial p}{\partial y}+\rho g$$

$$\frac{\partial P}{\partial x}=\frac{\partial p}{\partial x}$$

And as per the coordinate system, $g_y = -g$ & $g_x = 0$

Then the y component equation becomes,

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}-\rho g+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

x component equation becomes,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Since the system considered to be steady state, the term $\frac{\partial u}{\partial t}$ will be zero and the value of $g_x = 0$

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

If we are writing the order of magnitude analysis of the terms in the L.H.S of the equation

$$O\left(u\frac{\partial u}{\partial x}\right)\equiv\left(u_\infty.\frac{u_\infty}{L}\right)\equiv\left(\frac{{u_\infty}^2}{L}\right)$$

$$O\left(v\frac{\partial u}{\partial y}\right)\equiv\left(\frac{\delta}{L}.u_\infty.\frac{u_\infty}{\delta}\right)\equiv\left(\frac{{u_\infty}^2}{L}\right)$$

Both terms on L.H.S have same order of magnitude, $\left(\frac{{u_\infty}^2}{L}\right)$. So, both terms are equally important you cannot neglect any of the terms.

When we look at order of magnitude analysis of the stress or friction terms of R.H.S of the equation,

$$O\left(\frac{\partial^2 u}{\partial x^2}\right) = O\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right) = \frac{O(u)}{O(x)O(x)} = \frac{O(u)}{\left(O(x)\right)^2} = \frac{u_\infty}{L^2}$$

$$O\left(\frac{\partial^2 u}{\partial y^2}\right) = O\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\right) = \frac{O(u)}{O(y)O(y)} = \frac{O(u)}{\left(O(y)\right)^2} = \frac{u_\infty}{\delta^2}$$

We know that: $\delta \ll L \rightarrow \delta^2 \ll L^2 \rightarrow \frac{u_\infty}{\delta^2} \gg \frac{u_\infty}{L^2}$:from this we can write as,

$$O\left(\frac{\partial^2 u}{\partial y^2}\right) \gg O\left(\frac{\partial^2 u}{\partial x^2}\right)$$

Since this equation does not have genesis from mass balance, one of the stress or friction term $\left(\frac{\partial^2 u}{\partial x^2}\right)$ can be neglected w.r.t to $\left(\frac{\partial^2 u}{\partial y^2}\right)$.

In some other books its mentioned that due to boundary layer approximation $\left(\frac{\partial^2 u}{\partial x^2}\right)$=0.

Final equation of x component balance is:

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)$$

If we are considering the y component balance equation,

$$\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Since the system considered to be steady state, the term $\frac{\partial v}{\partial t}$ will be zero.

$$\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$O\left(u\frac{\partial v}{\partial x}\right) \equiv \left(u_\infty.\frac{u_\infty}{L}.\frac{\delta}{L}\right) \equiv \left(\frac{{u_\infty}^2}{L^2}.\delta\right)$$

$$O\left(v\frac{\partial v}{\partial y}\right) \equiv \left(\frac{u_\infty}{L}.\delta.\frac{u_\infty}{L}.\frac{\delta}{\delta}\right) \equiv \left(\frac{{u_\infty}^2}{L^2}.\delta\right)$$

Both the terms on the L.H.S of the equation have same order. If we are considering the skin friction terms of the R.H.S of the equation,

$$\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$O\left(\frac{\partial^2 v}{\partial x^2}\right) = O\left(\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\right) = \frac{O(v)}{O(x)O(x)} = \frac{O(v)}{\left(O(x)\right)^2} = \frac{O(v)}{L^2}$$

$$O\left(\frac{\partial^2 v}{\partial y^2}\right) = O\left(\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)\right) = \frac{O(v)}{O(y)O(y)} = \frac{O(v)}{\left(O(y)\right)^2} = \frac{O(v)}{\delta^2}$$

We know that: $\delta \ll L \to \delta^2 \ll L^2; \frac{O(v)}{L^2} \ll \frac{O(v)}{\delta^2}$,then we can neglect the term $O\left(\frac{\partial^2 v}{\partial x^2}\right)$ from the equation. After performing order of magnitude analysis, we will get y component balance equation as:

$$\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial y^2}\right)$$

Inertia terms→ $\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) \to \frac{{u_\infty}^2}{L}.\frac{\delta}{L}$

Pressure terms→ $\frac{1}{\rho}\frac{\partial P}{\partial y}$

Skin friction or stress term→ $\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial y^2}\right) \to \gamma\frac{\delta}{L}.\frac{u_\infty}{\delta^2}$

x component balance equation will be,

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)$$

Inertia terms→ $\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) \to \frac{{u_\infty}^2}{L}$

Skin friction or stress term→ $\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial y^2}\right) \to \gamma\frac{u_\infty}{\delta^2}$

Since $\frac{\delta}{L} \ll 1$,the inertia or the friction terms in the y component equation are actually lower than the corresponding terms in the x component equation.

If we take pressure friction from the order of magnitude analysis, order of pressure from this equation equal to the order of the skin friction terms in the x component equation. Similarly, the order of pressure is equal to the order of skin friction terms in the y component balance:

$$O\left(\frac{\partial P}{\partial x}\right) \approx \mu \frac{u_\infty}{\delta^2}$$

$$O\left(\frac{\partial P}{\partial y}\right) \approx \mu \frac{\delta}{L} \cdot \frac{u_\infty}{\delta^2}$$

From the order of magnitude analysis of continuity equation, we got flow within the boundary layer is 2D because x and y are both nonzero. Therefore,

$$P = P(x, y)$$

$$dP = \frac{\partial P}{\partial x} . dx + \frac{\partial P}{\partial y} . dy$$

$$= \mu \frac{u_\infty}{\delta^2} . L + \mu \frac{\delta}{L} \cdot \frac{u_\infty}{\delta^2} . \delta$$

$$= \left(\mu \frac{u_\infty}{L}\right) \frac{L^2}{\delta^2} + \mu \frac{u_\infty}{L}$$

Since $\frac{\delta}{L} \ll 1$,the term $\frac{L^2}{\delta^2}$ will be much higher. Therefore the $\left(\mu \frac{u_\infty}{L}\right) \frac{L^2}{\delta^2} \ggg \mu \frac{u_\infty}{L}$.So we can neglect the second term and this equation does not have its genesis in mass balance.

$$dP = \frac{\partial P}{\partial x} . dx \rightarrow \frac{dP}{dx} = \frac{\partial P}{\partial x}:$$

$$P = f(x)$$

that means, within the boundary layer P is a function of x only.

Thank you very much.