Chemical Engineering Fluid Dynamics and Heat Transfer
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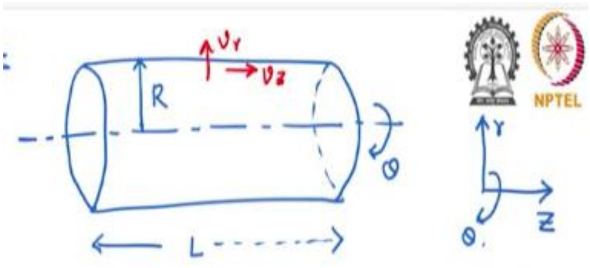
Lecture - 14
Exact Solution 4

Welcome back.

<u>Flow through a tube/Pipe</u>

Pressure driven flow:

Consider a pipe of uniform cross-sectional area with the length L and the radius R with the cylindrical coordinate system and listed below are the assumptions that we are going to consider resolving the problem.



Assumptions:

1.Incompressible fluid: $\rho \neq \rho(r, \theta, z, t)$

2.Steady state: All the time derivatives will be zero:$\frac{\partial}{\partial t}(\,) = 0$

3.Fully developed flow: $\left(\frac{\partial v_z}{\partial z}\right) = 0$:That means the flow in the z direction is fully developed. So, v_z does not change as a function of z.

4.system is θ symmetric:$\frac{\partial}{\partial \theta}(\,) = 0$:There is no variation in θ direction.

5.No slip condition is valid: At, r=R: v_z = 0 and v_θ=0 for all z and θ.

6.Solid impermeable wall: at, r=R; v_r =0 for all θ and z

Here we are taking cylindrical coordinate system to solve the problem r, θ,z and the component of velocities are in these three directions are v_r, v_θ, v_z respectively. So, how does the liquid flow through the pipe? The flow is a pressure driven flow.

Next, we will consider the governing equation in cylindrical coordinate system.

Continuity equation in r- θ - z system

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Navier 's equation:

r- component equation:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial r} + \rho g_r - \left(\frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right)$$

θ- component equation

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta - \left(\frac{1}{r^2}\frac{\partial (r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right)$$

z- component equation

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z - \left(\frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right)$$

So, we will start the analysis of continuity equation,

Continuity equation in r- θ - z system

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Based on the assumption that incompressible fluid, the first term ($\frac{\partial \rho}{\partial t}$) will be zero. As the system being θ symmetric, this term $\frac{\partial}{\partial \theta}(\rho v_\theta)$ also will be zero. Since the flow is fully developed flow, we can say that $\frac{\partial}{\partial z}(\rho v_z) = 0.$

The term remaining in the continuity equation after simplification,

$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) = 0;$

Partial differential equation becomes ordinary differential equation,

$\frac{1}{r}\frac{d(\rho r v_r)}{dr} = 0;$

Upon integration, $(r v_r) = C_1$

From the boundary conditions: for r=R, $v_r = 0$

$$(R v_r) = 0 = C_1$$

$v_r = 0$ for the flow field.

Next, we will start analyzing the r component balance equation.

$$\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial r} + \rho g_r - \left(\frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right)$$

As per assumption steady state system, first term ($\frac{\partial v_r}{\partial t}$) will be zero. And also, from the continuity analysis we got $v_r = 0$,it makes the second term $v_r \frac{\partial v_r}{\partial r}$ zero. Since the system is θ symmetric, the term $\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}$ will be zero. And there is no flow in the radial direction therefore, $\frac{v_\theta}{r}$ will be zero. Because of the assumption fully developed flow the term $v_z \frac{\partial v_z}{\partial z}$ will be zero. Then all the terms on the L.H.S of the r component balance equation are zero.

If we look at the R.H.S of the r component balance equation, the first term $\frac{\partial P}{\partial r}$ will be non-zero.

we knew that $\tau_{rr} = f\left(\frac{\partial v_r}{\partial r}\right)$,Since $v_r = 0$ the term $\tau_{rr} = 0$ and it makes the whole term $\frac{1}{r}\frac{\partial(r\tau_{rr})}{\partial r}$ as zero. Similarly, $\tau_{\theta\theta} = f\left(\frac{\partial v_\theta}{\partial r}\right)$ since we assume that system is θ symmetric v_θ=0: $\tau_{\theta\theta} = f\left(\frac{\partial v_\theta}{\partial r}\right) = 0 \rightarrow \frac{\tau_{\theta\theta}}{r} = 0$.and also the term $\frac{\partial \tau_{r\theta}}{\partial \theta}$ will be zero due to the θ symmetry.

In this, the term $\frac{\partial \tau_{rz}}{\partial z} = \frac{\partial}{\partial z}\left(f\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)\right)$: $\frac{\partial v_r}{\partial z} = 0$ due to v_r=0 :then the term finally becomes $\frac{\partial}{\partial z}\left(f\left(\frac{\partial v_z}{\partial r}\right)\right)$ and the that term can be written as

$\frac{\partial}{\partial z}\left(f\left(\frac{\partial v_z}{\partial r}\right)\right) = f\left(\frac{\partial}{\partial r}\left(\frac{\partial v_z}{\partial z}\right)\right)$ and we know that this term will get to zero due to the fully developed flow condition.

And we are taking the direction of gravity in the downwards direction, therefore the term, $g_r = g$.

the r component balance equation becomes,

$$-\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{r}} + \boldsymbol{\rho g} = \mathbf{0}$$

We are considering the non-gravitational pressure,

$$P = p - \rho g r$$

Upon differentiation: $\frac{\partial P}{\partial r} = \frac{\partial p}{\partial r} - \rho g$: If we combine the equations, we will get $\frac{\partial P}{\partial r} = 0$

And we can say that $\frac{\partial P}{\partial \theta} = \frac{\partial p}{\partial \theta}$ and $\frac{\partial P}{\partial z} = \frac{\partial p}{\partial z}$

Since $\frac{\partial P}{\partial r} = 0$, we can write as $P \neq f(r)$ and due to θ symmetric, $\frac{\partial P}{\partial \theta} = 0$: $P \neq f(\theta)$

It shows that $\boldsymbol{P = f(z)}$ only.

From the θ component balance equation,

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta - \left(\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right)$$

Similar to r component balance equation, the term $\frac{\partial v_\theta}{\partial t}$ will be zero due to the steady state assumption. Since $v_r = 0$,the term $v_r \frac{\partial v_\theta}{\partial r}$ and the term $\frac{v_r v_\theta}{r}$ becomes zero. Due to the θ symmetry, the term $\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}$ and the term $v_z \frac{\partial v_\theta}{\partial z}$ will be zero.

If we look into R.H.S of the equation of the θ component balance equation,

As per the definition, $\tau_{r\theta} = f\left(\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r}\right)$

Since v_r and v_θ are zero, this term $\frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r}$ also become zero. Since the system is θ symmetric the term $\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}$ will be zero. Similar to $\tau_{r\theta}$, $\tau_{\theta z} = f\left(\frac{\partial v_\theta}{\partial z} + \frac{\partial v_z}{\partial \theta}\right)$ and this term also becomes zero. And g does not act in the direction of θ.therefore, $g_\theta = 0$.

The only one term remaining in the θ component balance equation,

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = 0:$$

$$\boldsymbol{\frac{\partial P}{\partial \theta} = 0}$$

If we look into the z- component equation

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z - \left(\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right)$$

the term $\frac{\partial v_z}{\partial t}$ will be zero due to the steady state assumption. Since $v_r = 0$,the term $v_r \frac{\partial v_z}{\partial r}$ will be zero. The term $\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}$ also zero due to the θ symmetry. The last term of the L.H. S of the z component balance equation $v_z \frac{\partial v_z}{\partial z}$ is zero due to the fully developed flow. If we consider the R.H.S of the equation,

The term $\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} = 0$ due to the θ symmetry. And we knew that $\tau_{zz} = f\left(\frac{\partial v_z}{\partial z}\right)$ and this term will be zero due to the fully developed flow assumption. The term, ρg_z also zero, because g does not act in the direction of z.

From the z- component balance equation we will get,

$$-\frac{\partial p}{\partial z}-\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r}=0$$

$$-\boldsymbol{\frac{\partial P}{\partial z}=\frac{1}{r}\frac{d(r\tau_{rz})}{dr}}$$

$\frac{dP}{dz}$ is the pressure drop along the length of the tube. And if the volumetric flow rate is constant and diameter of the tube is constant then at every section $\frac{dP}{dz}$ is actually constant.

$\frac{dP}{dz} = constant{=}C_2$

Upon integration, $\int_{P_1}^{P_2} dP = C_2 \int_{Z_1}^{Z_2} dz$

$$\Delta P = C_2(z_2 - z_1): C_2 = \frac{\Delta P}{L}$$

Negative sign actually shows that as you flow along the direction the pressure reduces.

$$-\frac{\Delta P}{L}=\frac{1}{r}\frac{d(r\tau_{rz})}{dr}$$

Upon integration,

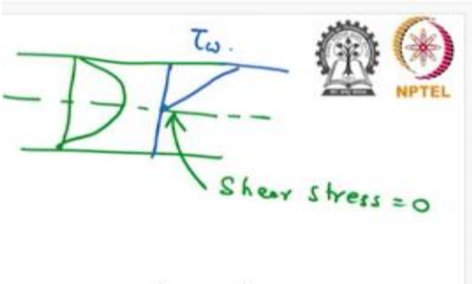
$$r\tau_{rz} = \left(-\frac{\Delta P}{L}\right)\frac{r^2}{2} + C_3$$

From the boundary condition, we know that velocity is going to be maximum at the center and shear stress is minimum, that means zero and at the center, r=0 shear stress will be zero. So the boundary conditions are,

at r=0, $\tau_{rz} = 0\colon C_3 = 0$

Then the shear stress: $\tau_{rz} = \left(-\frac{\Delta P}{2L}\right) r$: that the shear stress profile is linear, and its value is linearly increases from 0 to the maximum value, $\tau_w = \tau_{rz}|_{r=R}$.

Wall shear stress: $\tau_w = \tau_{rz}|_{r=R} = \left(-\frac{\Delta P}{2L}\right).R$



Velocity profile for the power law fluid:

For a power law of fluid, the shear stress dependence in a tube flow is given as

$$\tau_{rz} = m\left(-\frac{dv_z}{dr}\right)^n$$

$$\left(-\frac{dv_z}{dr}\right) = \left[\frac{r}{2m}\left(\frac{-\Delta P}{L}\right)\right]^{\frac{1}{n}}$$

Upon integration $$-v_z = \left(\frac{-\Delta P}{2mL}\right)^{\frac{1}{n}}\left[\frac{r^{\frac{n+1}{n}}}{\frac{n+1}{n}}\right] + C_4$$

To evaluate C_4, Boundary conditions at r=R, $v_z = 0$ →Then the $C_4 = -\left(\frac{-\Delta P}{2mL}\right)^{\frac{1}{n}}\frac{n}{n+1}\left[R^{\frac{n+1}{n}}\right]$

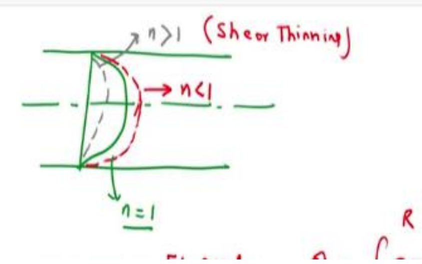
$$v_z = \left(\frac{-\Delta P}{2mL}\right)^{\frac{1}{n}}\frac{n}{n+1}\left[R^{\frac{n+1}{n}}\right]\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

In the case of Newtonian's fluid: $m = \mu$ and $n = 1$

Then the equation

$$v_z = \left(\frac{-\Delta P}{2\mu L}\right)R^2\left[1-\left(\frac{r}{R}\right)^2\right]$$

From the above equation, it is clear that, when the Newtonian fluid passes through a tube, it will have a parabolic profile. The velocity profile equation for a Newtonian fluid is called the Hagen Poiseuille equation or Hagen Poiseuille flow. For shear thinning fluid,$n > 1$,it shows in the image.



Volumetric flow rate

$$Q = \int_0^R 2\pi r . v_z . dr$$

$$= \int_0^R 2\pi r \left(\frac{-\Delta P}{2mL}\right)^{\frac{1}{n}} \frac{n}{n+1} \left[R^{\frac{n+1}{n}}\right] \left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] dr$$

At r=0, Velocity will be maximum: $v_{zmax} = v_z|_{r=0}$

$$v_{zmax} = \left(\frac{-\Delta P}{2mL}\right)^{\frac{1}{n}} \frac{n}{n+1} \left[R^{\frac{n+1}{n}}\right]$$

Then the volumetric flow rate will be

$$Q = \int_0^R 2\pi r \; v_{zmax} \left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] dr$$

$$Q = \pi \, v_{zmax} \left(\frac{n+1}{3n+1}\right) R^2$$

Next class we will discuss about boundary layer concept.Thank you.