Chemical Engineering Fluid Dynamics and Heat Transfer
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Lecture - 12
Exact Solution 2

So, welcome back to the exact solutions. So, we started to talk about the possible Exact Solution of the Navier-Stokes equation or more critically the Navier's equation for a fluid flowing over an inclined plane.

x component momentum balance equation:

we will start with x component momentum balance equation.

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+\rho g_x$$

Since the flow is steady state flow, $\frac{\partial u}{\partial t}=0$ and also the flow is full developed flow the term $u\frac{\partial u}{\partial x}$ will be zero. And due to v=0 and w=0, the other two terms on the L.H. S of the equation becomes zero. Collectively all the terms on the L.H.S of the x component momentum balance equation will be zero. If we are considering the R.H.S of the equation,

$$\frac{\partial \tau_{xz}}{\partial z}=f\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0$$

$$\frac{\partial \tau_{xy}}{\partial y}=f\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\neq 0$$

$$\frac{\partial \tau_{xx}}{\partial x}=f\left(\frac{\partial u}{\partial x}\right)=0$$

Then upon simplification the z component balance equation becomes

$$\frac{\partial \tau_{xy}}{\partial y}=-\frac{\partial p}{\partial x}+\rho g_x$$

Due to the orientation of the coordinate axis from the image, the value of $g_x=-g\sin\theta$

Therefore, upon simplification the eqn becomes,

$$\frac{\partial \tau_{xy}}{\partial y} = -\frac{\partial p}{\partial x} - g \sin\theta$$

Note: Fully developed flow does not mean that all variations in the x direction will be zero. Only the velocity in the x direction will be zero. Therefore, the pressure gradient will not be zero in case of a pressure driven flow.

For y component momentum balance equation, we got.

$$\frac{\partial p}{\partial y} = -\rho g \cos\theta$$

We know that $p = p(x, y, z, t)$

And from the steady state and incompressible flow assumptions $p \neq f(t)$ and also from the z component momentum balance equation $\frac{\partial p}{\partial z} = 0$,

Then we can say that $P = P(x, y)$:

At any particular x, $P(x) =$ pressure at a particular x becomes function of y only.

x=x_1: $P(x_1) = f(y)$only

then the above partial differential equation can be written as

$$\frac{dP(x)}{dy} = -\rho g \cos\theta$$

If we integrate the above equation we will get,

$$P(x) = -\rho g \cos\theta y + C_1$$

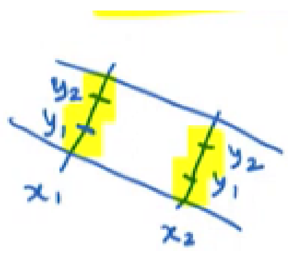
By using the Boundary conditions

For y=H, $P(x) = P_{atm}$: $\quad P_{atm} + \rho g \cos\theta H = C_1$

Therefore $\quad P(x) = P_{atm} + \rho g \cos\theta (H - y)$

If we at $(x_1 y_1)$: $\quad P(x_1) = P_{atm} + \rho g \cos\theta (H - y_1)$

If we at $(x_2 y_1)$: $\quad P(x_2) = P_{atm} + \rho g \cos\theta (H - y_1)$



Therefore $\Delta P(x)|_{y_1}$= 0: that means $\Delta P(x)$ for all y=0

For all values of y: $\frac{dP(x)}{dx} = 0 \rightarrow \frac{\partial P}{\partial x} = O$ that means $P \neq f(x)$; P=f(y) only.

So, the equation for the pressure distribution in the flow over the inclined plane can be written as $P = P_{atm} + \rho g \cos\theta\,(H - y)$

From the equation, it is clear that as the depth decreases the pressure increases. So, this implies that the pressure distribution is purely hydrostatic.

At the surface of the liquid layer, y=H: $P = P_{atm}$

At the surface of the plane, i.e., y=0: $P = P_{atm} + \rho g \cos\theta\, H$

By substituting $\frac{\partial P}{\partial x} = 0$ to the simplified x component equation

$$\frac{d\tau_{xy}}{dy} = -\rho g \sin\theta$$

$$\tau_{xy} = -\rho g \sin\theta y + C_2$$

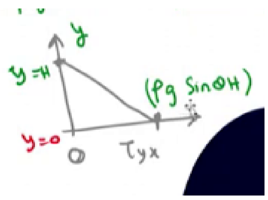
Boundary Conditions to evaluate C_2 : Shear stress at the free surface is 0.

at $y = H, \tau_{xy} = 0 \rightarrow C_2 = \rho g \sin\theta\, H$

Then the shear stress distribution equation becomes,

$$\tau_{xy} = \rho g \sin\theta (H - y)$$

Linear shear stress distribution within the flow of fluid over an inclined plane.



And at y=0,$\tau_{xy} = \rho g \sin\theta H$ Shear stress will be maximum at the surface of the wall and at y=H, $\tau_{xy} = \rho g \sin\theta (H - y) = 0$,shear stress will be zero at the free surface. Shear stress profile is linear in this example. This expression of shear stress of a fluid flowing over an inclined plane is independent of the nature of the fluid. So, this shear stress profile is valid for all types of fluids.

For Newtonian fluid: $\tau_{yx} = \mu \frac{du}{dy}$:From this expression you will get the velocity profile.

For in the case of power law fluid,: $\tau_{yx} = m \left(\frac{du}{dy}\right)^n$

If we substitute this equation in the shear stress distribution equation

$$\tau_{yx} = m \left(\frac{du}{dy}\right)^n = \rho g \sin\theta (H - y)$$

$$\left(\frac{du}{dy}\right) = \left[\frac{\rho g}{m} \sin\theta (H - y)\right]^{\frac{1}{n}}$$

If we are integrating the above equation,

$$u = \left(\frac{\rho g \sin\theta}{m}\right)^{\frac{1}{n}} \int_0^H (H - y)^{\frac{1}{n}}\ dy$$

Let's assume $H - y = \eta$: $dy = -d\eta$ to perform the integral if we substitute this in above equation. Then the equation becomes,

$$u = -\left(\frac{\rho g \sin\theta}{m}\right)^{\frac{1}{n}} \int_0^H (\eta)^{\frac{1}{n}}\ d\eta$$

Upon integration the equation becomes,

$$u = -\left(\frac{n}{n+1}\right)\left(\frac{\rho g \sin\theta}{m}\right)^{\frac{1}{n}} \eta^{\frac{n+1}{n}} + C_3$$

To evaluate C_3, we will use the no slip boundary condition:

1)no slip boundary condition at y=0: u=0→ $\eta = H$:u=0

By applying the boundary condition we will get,

$$C_3 = \left(\frac{n}{n+1}\right)\left(\frac{\rho g \sin\theta}{m}\right)^{\frac{1}{n}} H^{\frac{n+1}{n}}$$

If we put the value of C_3 in the above equation

$$u = \left(\frac{n}{n+1}\right)\left(\frac{\rho g \sin\theta}{m}\right)^{\frac{1}{n}}\left[H^{\frac{n+1}{n}} - (H-y)^{\frac{n+1}{n}}\right]$$

$$= \left(\frac{n}{n+1}\right)\left(\frac{\rho g \sin\theta}{m}\right)^{\frac{1}{n}} H^{\frac{n+1}{n}}\left[1 - \left(1 - \frac{y}{H}\right)^{\frac{n+1}{n}}\right]$$

Volumetric flow rate is the product of average velocity and the cross-sectional area. From the volumetric flow rate, we can find out the average velocity.

<u>Velocity profile for a Bingham plastic:</u>

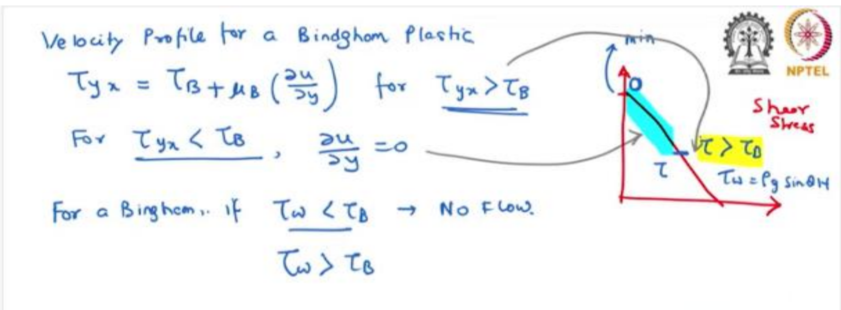
Bingham plastic is a type of fluid, for which it does not deform initially when we applied force and once the applied stress crosses the yield stress or the critical stress then it behaves like a Newtonian fluid.

For $\tau_{yx} > \tau_B$: $\tau_{yx} = \tau_B + \mu_B\left(\frac{\partial u}{\partial y}\right)$

For $\tau_{yx} < \tau_B$; $\frac{\partial u}{\partial y} = 0$

And we knew that shear stress will be minimum at the free surface and maximum at the wall (τ_w).

If $\tau_w < \tau_B$:There will not be any flow. Once the τ_w exceeds the τ_B ,it starts to flow.



We will continue the discussion in the next class.

Thank you.