Chemical Engineering Fluid Dynamics and Heat Transfer

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Lecture - 11
Exact Solution 1

Welcome back to this lecture, Exact Solutions. So, far we have understood the conservation equations that means, conservation of mass, conservation of momentum for a flowing for a fluid and we also understood the effects of the stresses how they manifest in conservation of momentum.

The momentum balance equation in terms of shear stresses is known as the Navier's equation or the Cauchy's equation. If we represent the shear stresses in terms of velocity gradient for a Newtonian fluid, then it is known as Navier Stokes equation. The advective terms on the left-hand side of the Navier Stokes equation makes the equation nonlinear. Therefore, it is difficult to solve in most cases. Navier Stokes equation does not have an analytical solution. However, there are special cases based on certain assumptions or physical conditions it becomes possible to solve the Navier Stokes equation and those are called the exact solutions. Here we are looking into the few examples.

<u>Flow of a liquid over an inclined plane:</u>

It is a gravity driven flow of liquid over an inclined plane. In fact, this is very similar to flow of river or dip coating process. So, here essentially the problem statement is to develop the general equation of flow of a liquid over an inclined plane.

Let us begin with Navier's equation:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+\rho g_x$$

In the above Navier's equation, τ_{xx} refers to the deviatoric component of the normal stress and if we are writing the Navier's equation in terms of total normal stress the equation becomes,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+\rho g_x$$

σ_{xx} represents the total normal stress which includes both deviatoric component of normal stress and hydrostatic part of normal stress. This above equation in terms of stresses is called Navier's equation or Cauchy's equation.

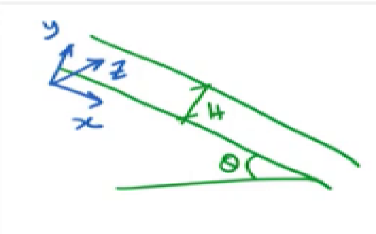
For a Newtonian fluid, Shear stress $\tau_{xy} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$,So if we substitute these terms in the Navier's equation, the equation becomes

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

This form of the above equation is known as the Navier Stokes equation. NS equation is valid for the stokes fluid.

$\tau_{xy} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ →this expression is valid for a Stokes fluid. Newtonian fluid is a type of stokes fluid. So, we cannot use the Navier Stokes equation for a power law fluid, since the stokes hypothesis is not valid for a power law fluid.

So, here we come back to the problem, a general equation of flow of a fluid over an inclined plane. So, we will start with the Navier's equation for analysis.



Consider an inclined plane having an angle θ with the horizontal and liquid layer of thickness, H on the top with the coordinate system as shown in the figure. First, we will look into the assumptions.

1.Steady state flow: All the time derivatives will be zero:$\frac{\partial}{\partial t} = 0$

2.Incompressible flow; $\rho \neq \rho(x, y, z, t)$

3.There is no flow in z direction, i.e., velocity in z direction will be zero, w=0

4.The z direction is very wide and therefore there is no variation of properties in the z direction.$\left(\frac{\partial}{\partial z}\right) = 0$

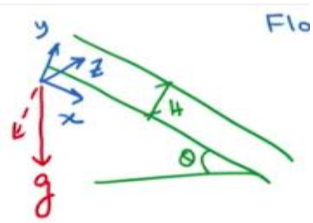
5.Fully developed flow,$\left(\frac{\partial u}{\partial x}\right) = 0$:That means the flow in the x direction is fully developed. So, u does not change as a function of x.

6.Shear stress will be zero at the free surface.

7.No slip condition is valid at the wall: At y=0: u =0 and w=0 for all x and z

8.Solid impermeable wall: At y=0; v=0 for all x and z

for example, if there is a flow over a porous surface or membrane, there will be nonzero value of v.



we now do the analysis of the four constituent governing equations or the conservation equations, which are the continuity and three components of the velocity.

<u>Equation of continuity</u>

(Since the fluid is incompressible, we will use this form of conservation of mass)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

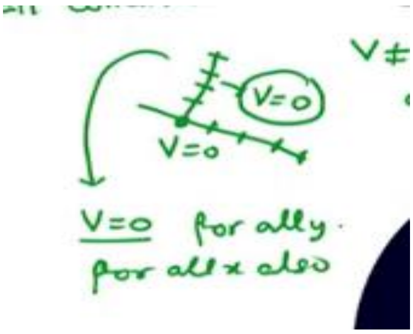
Since the flow is fully developed flow, the first term, $\frac{\partial u}{\partial x}$ =0 and there is no variation in the z direction (assumption no 3), the term $\frac{\partial w}{\partial z} = 0$

Then we will get: $\frac{\partial v}{\partial y} = 0 \rightarrow v \neq f(y)$: $v = f(x, z)$;Since there is no variation in the z direction,$\frac{\partial v}{\partial z} = 0$: then we can say that $\boldsymbol{v} = \boldsymbol{f}(\boldsymbol{x})$ only.

Because of the solid impermeable wall condition at y=0; v=0 for all x and z and from the continuity equation we got $v \neq f(y)$.That means v does not change with y. So, if at one point the value is 0, then at all other points the value of v will be 0. then we can say that all points of x and y, v = 0.

So, therefore, within the entire flow field

$$v = 0.$$



<u>z component momentum balance</u>

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}+\rho g_z$$

Since the flow is steady state flow, $\frac{\partial w}{\partial t}=0$

There is no variation or flow in the z direction, therefore w=0 and $\left(\frac{\partial}{\partial z}\right)=0$:it makes the terms $u\frac{\partial w}{\partial x}=0$ and $w\frac{\partial w}{\partial z}=0$, $\frac{\partial \tau_{zz}}{\partial z}=0$ [assumption no 4:]

And from the continuity equation analysis we got v=0, therefore the term $v\frac{\partial w}{\partial y}=0$

Collectively all the terms on the L.H.S of the equation will be zero. And if we are considering the terms on the R.H.S of the equation, as per the assumptions these terms will also be zero

$$\frac{\partial \tau_{xz}}{\partial x}=f\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0$$

$$\frac{\partial \tau_{yz}}{\partial y}=f\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)=0$$

$$\frac{\partial \tau_{zz}}{\partial z} = f\left(\frac{\partial w}{\partial z}\right) = 0$$

And there is no coordinate axes of gravity acts in the direction of z, therefore the term $\rho g_z = 0$

Finally, you will get $\boldsymbol{\frac{\partial p}{\partial z} = 0}$

Note: we can say that (assumption no 4) there is no variation along with z direction, therefore the term can be $\frac{\partial p}{\partial z} = 0$

<u>y component momentum balance</u>

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho g_y$$

Similar z component momentum balance,

Since the flow is steady state flow, $\frac{\partial v}{\partial t} = 0$

Due to v=0 and w=0, all the three terms on the L.H.S of the above equation becomes zero. According to the assumption all these terms on the R.H.S of the equation will be zero.

$$\frac{\partial \tau_{xy}}{\partial x} = f\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\frac{\partial \tau_{yz}}{\partial z} = f\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

$$\frac{\partial \tau_{yy}}{\partial y} = f\left(\frac{\partial v}{\partial y}\right) = 0$$

Therefore, the equation becomes $\boldsymbol{-\frac{\partial p}{\partial y} + \rho g_y = 0}$

From the figure, $g_y = -g\cos\theta$

Therefore, the z component equation becomes

$$\frac{\partial p}{\partial y} = -\rho g\cos\theta$$

Next class we will discuss the x component momentum balance equation. Thank you