Chemical Engineering Fluid Dynamics and Heat Transfer
Prof. Rabibrata Mukherjee
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 10
Conservation Equation 05 - Conservation of Momentum - 03

Welcome back.

In the last lecture, we have discussed about the momentum balance. We investigated how the different components of momentum enters the control volume. x component momentum enters the left face of the control volume due to the x component velocity. And also, it enters the front face of the control volume due to the y component velocity and through the bottom face of the control volume because of the z component velocity. Then we looked into how the surface forces, or the stresses are acting on a control volume. In other words what are the places, or which are the surfaces are subject to stresses due to the x component velocity.

And what we understood is that on the left face x component velocity exerts normal stress and then the x component velocity since it is passing over the front face and the rear face and if there is a variation in the velocity then that will try to cause a deformation in a representative x y plane of the control volume that is passing through the centroid.

So,because of this velocity gradient, gradient in the x component velocity along the y direction it will try to deform an x y plane or an x y plane on the control volume experiences the stress therefore, this contributes to the x y stress or τ_{xy}.

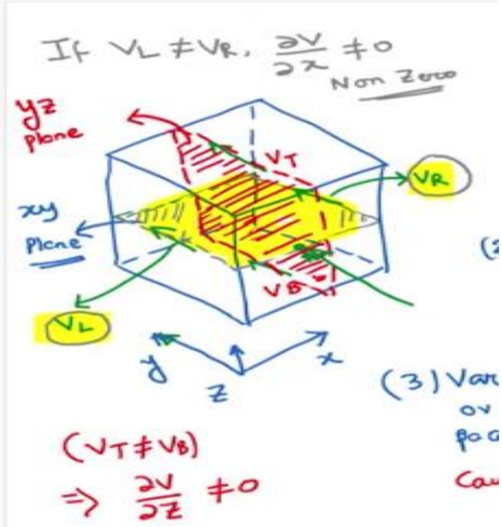
There is also x component velocity or u flows over the top and the bottom face and if there is a variation of u as a function of z ,this tries to cause a deformation in an x z plane and therefore, there is active τ_{xz}which also contributes into the forces or the stresses that get generated due to the x component velocity . So, based on these concepts we got this functional form of the momentum balance equation,

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}\right)$$

This equation is known as the **Navier's equation or the Cauchy's equation**.

The advantage of this **Navier's equation or the Cauchy's equation** is that,it is independent of the nature of the fluid. So, whether you have a Newtonian fluid or you have a Bingham or a non-Newtonian fluid like power law fluid, we can use this equation.

We will consider a control volume and we will look into the stresses caused by the y component velocity.



1.On the font face, v causes normal stress σ_{xx}.

2.variation of value of V over the left and right face$\left[\left(\frac{\partial v}{\partial y}\right) \neq 0, v = f(y)\ \right]$,it leads to τ_{xy} stress acting on the left and right faces.

3. if there is a variation in the value of V over the bottom and the top $\left(\frac{\partial v}{\partial z}\right) \neq 0$ is going to cause τ_{yz}.

If there is no velocity gradient, then there will be no tendency of deformation. Therefore

$V_L \neq V_R, \frac{\partial V}{\partial x} \neq 0$(Non zero).similarly, $V_T \neq V_B, \frac{\partial V}{\partial Z} \neq 0$

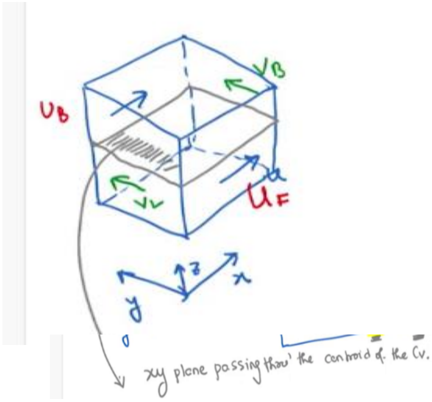
The y component velocity exerts normal stress on the front face, but in addition because of the flow of the y component velocity on the over the left and the right phase and if there is a variation between them and they actually cause τ_{xy}or as this variation tries to deform xy plane. So, therefore, $\tau_{xy}$is active over on the left and the right face and in very similar manner if there is a variation of V with z that is V_T and V_B are different then it tries to deform yz plane and in fact, that causes τ_{yz} due to y component velocity.

Next, we are looking into the x component velocity acts on the xy plane passing though the centroid. As we discussed before,

$\frac{\partial u}{\partial y} \neq 0$:due to $u_F \neq u_B$.it tries to cause the deformation of xy plane (due to the x component velocity)

$V_L \neq V_B, \frac{\partial V}{\partial x} \neq 0$: It tries to cause deformation into xy plane (due to the y component velocity)

The total shear stress that is acting on the plane has two components. Therefore,total angular deformation happens to the xy plane is:



$$\tau_{xy} = f\left(\frac{\partial V}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Next, we will consider the Navier's equation or the Cauchy's equation.

$$\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right)$$

This equtaion is not mathematically closed, because of the fact that it brings in new terms like $\sigma_{xx}, \tau_{xy}, \tau_{xz}$. So, this actually brings in more variables than the number of equations. So, we can split the net stress into two parts: one related to deformation which is known as the deviatoric component and the other independent of deformation which is the hydrostatic component.

$$\tau_{ij} = \tau_{ij}^{deviatoric} + \tau_{ij}^{Hydrostatic}$$

Angular deformation is caused by terms like $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial z}$ etc. So, any generic term can write,

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{Symmetric part (Angular deformity)}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{Anti Symmetric part (Rotation)}}$$

Symmetric part (Angular deformity) | Anti Symmetric part (Rotation)

Mathematically, the first term is called the symmetric part and the second term is called the anti-symmetric part, but symmetric part is related to angular deformation and anti symmetric part is related to rotation. And we know that rotation does not cause any shear stress. Therefore, we can say that

$$\tau_{ij}^{deviatoric} = f(rate\ of\ angular\ deformation)$$

Based on a rigorous mathematical derivation leads to

$$\tau_{ij}^{dev} = \lambda. div\, \vec{V} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

For an incompressible fluid, $div\, \vec{V} = 0$,then the expression for deviatoric stress becomes

$$\tau_{ij}^{dev} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$\lambda \rightarrow 2nd\ coefficient\ of\ viscosity$

For a Stoke's fluid:

$$K = \lambda + \frac{2}{3}\mu = 0$$

$\mu \rightarrow coefficient\ of\ viscosity: K \rightarrow coefficient\ of\ bulk\ viscosity$

A Stokes fluid is a fluid for which the thermodynamic pressure is equal to mechanical pressure. Thermodynamic pressure is the pressure due to the total energy of the all the molecule, and the mechanical pressure is due to the kinetic energy of the molecule.

For stokes fluid,$P_T = P_M$

Kronekar delta: $\delta_{ij} = 1: if i = j\ or\ \delta_{ij} = 0: if i \neq j$

Note: It means that it is valid only for normal stress.

The hydrostatic component the expression,

$$\tau_{ij}^{Hydrostatic} = -p\delta_{ij}$$

So, we can write for the normal stress:

$$\sigma_{xx} = -p + \tau_{xx}$$

Note: So, the hydrostatic component only gets active for a normal component of the stress and this component is active even if the fluid is not moving. So, basically the normal stress is going to be higher and that is due to the hydrostatic component