# Advance Process Dynamics

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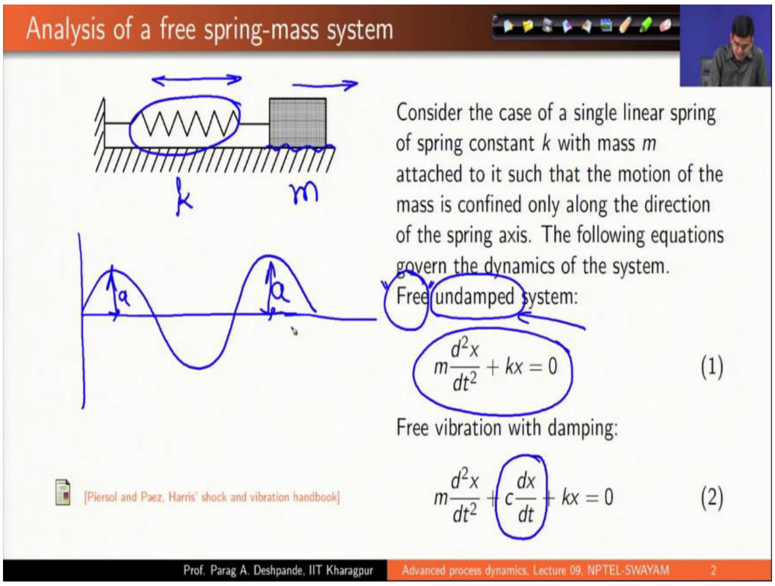
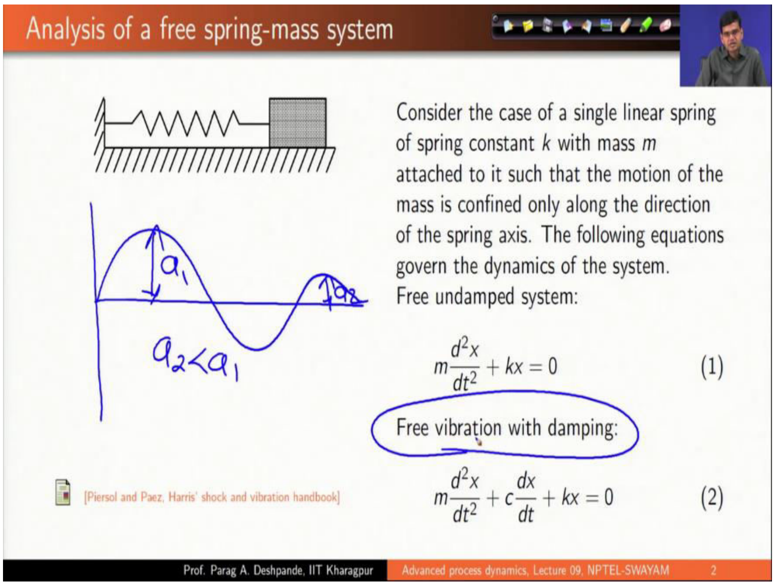
Lecture 09

## Analysis of a free spring-mass system



## Analysis of a free spring-mass system

Convert the dynamical equations into matrix equations and analyse

- the equilibrium solution(s)
- the phase portraits
- the stability of the system
- the effect of different parameters on the dynamical behaviour of the system

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{dx}{dt} = y \quad -(1)$$

$$m\frac{dy}{dt} + kx = 0 \quad -(2)$$

$$\frac{dx}{dt} = 0x + 1y$$

$$\frac{dy}{dt} = -\frac{k}{m}x + 0y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$



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- the equilibrium solution(s)
- the phase portraits
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- the effect of different parameters on the dynamical behaviour of the system

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

$$\frac{dx}{dt} = y \quad -(1)$$

$$m\frac{dy}{dt} + cy + kx = 0 \quad -(2)$$

$$\frac{dx}{dt} = 0x + 1y$$

$$\frac{dy}{dt} = -\frac{k}{m}x - \frac{c}{m}y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

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## Analysis of a free spring-mass system

$$m\frac{d^2x}{dt^2} + kx = 0$$

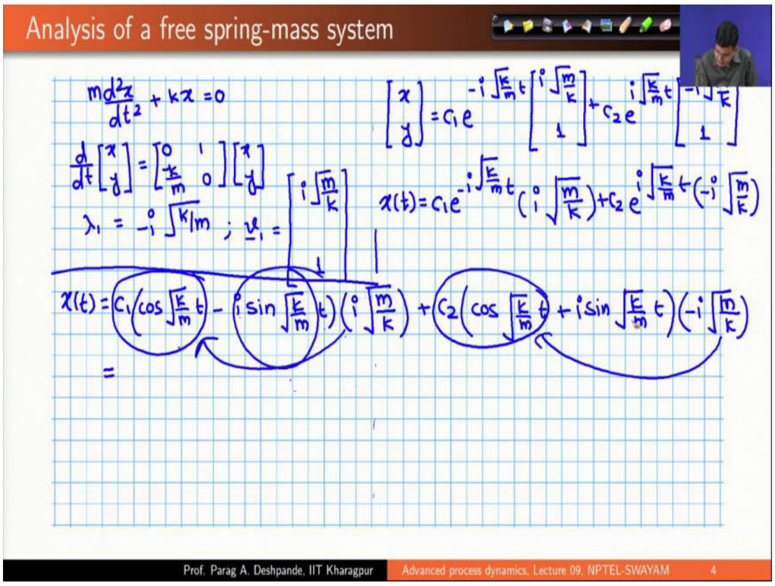
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_1 = -i\sqrt{k/m} \; ; \; \underline{v}_1 = \begin{bmatrix} i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}$$

$$\lambda_2 = i\sqrt{\frac{k}{m}} \; ; \; \underline{v}_2 = \begin{bmatrix} -i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-i\sqrt{\frac{k}{m}}t}\begin{bmatrix} i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix} + c_2 e^{i\sqrt{\frac{k}{m}}t}\begin{bmatrix} -i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}$$

$$x(t) = c_1 e^{-i\sqrt{\frac{k}{m}}t}\left(i\sqrt{\frac{m}{k}}\right) + c_2 e^{i\sqrt{\frac{k}{m}}t}\left(-i\sqrt{\frac{m}{k}}\right)$$


Analysis of a free spring-mass system
m d²x/dt² + kx = 0
d/dt [x y] = [0 1; -k/m 0] [x y]
λ₁ = -i √(k/m) ; v₁ = [i √(m/k) 1]
[x y] = c₁ e^(-i√(k/m) t) [i√(m/k) 1] + c₂ e^(i√(k/m) t) [-i√(m/k) 1]
x(t) = c₁ e^(-i√(k/m) t) (i√(m/k)) + c₂ e^(i√(k/m) t) (-i√(m/k))
x(t) = c₁(cos √(k/m) t - i sin √(k/m) t)(i √(m/k)) + c₂(cos √(k/m) t + i sin √(k/m) t)(-i √(m/k))
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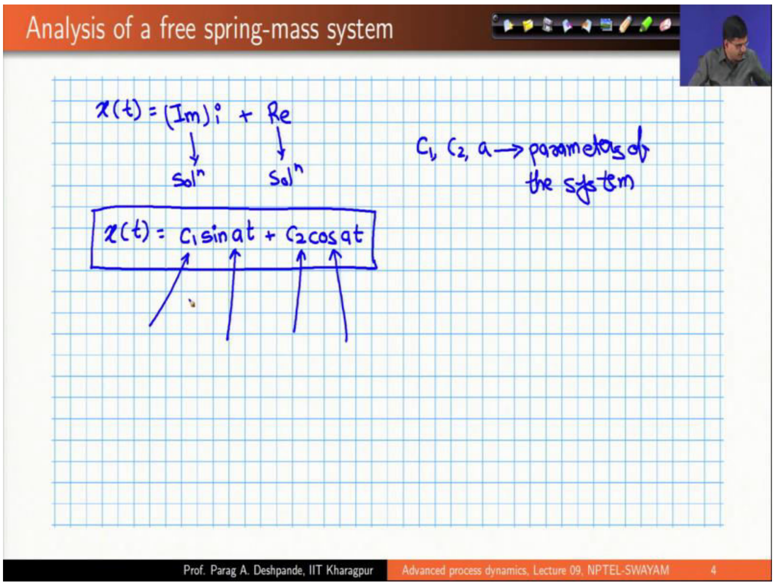
Analysis of a free spring-mass system
x(t) = (Im) i + Re
Soln
Soln
c₁, c₂, a → parameters of the system
x(t) = c₁ sin at + c₂ cos at
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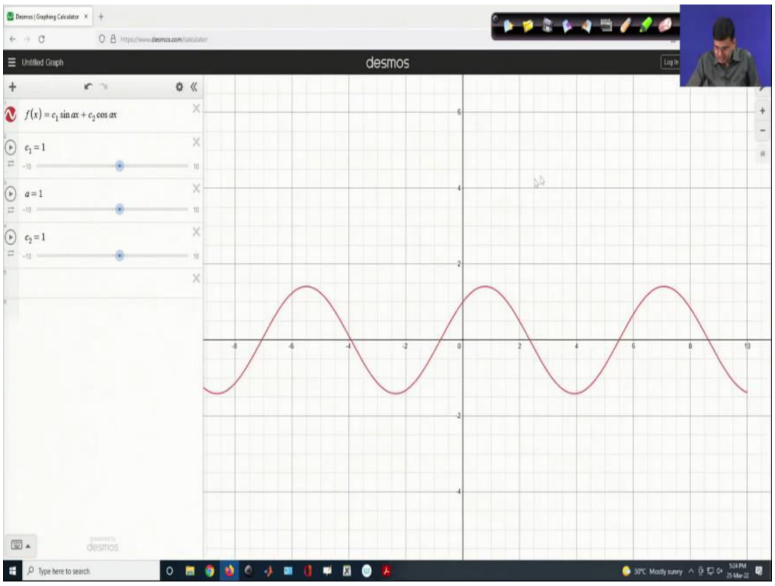
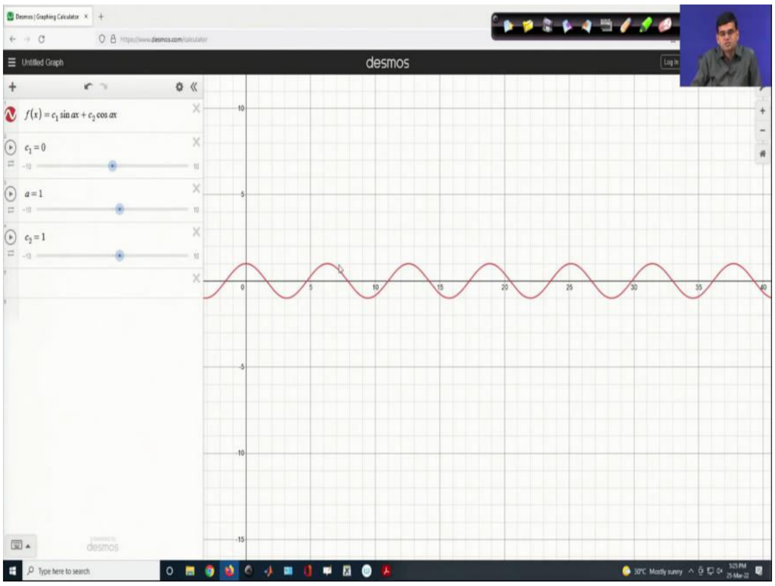
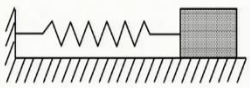
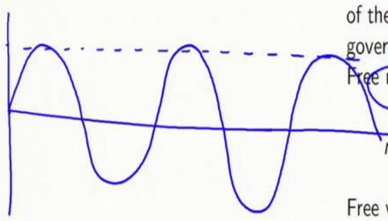

desmos
Untitled Graph
f(x) = c₁ sin ax + c₂ cos ax
c₁ = 1
a = 1
c₂ = 1




## Analysis of a free spring-mass system





Consider the case of a single linear spring of spring constant $k$ with mass $m$ attached to it such that the motion of the mass is confined only along the direction of the spring axis. The following equations govern the dynamics of the system.

Free undamped system:

$$m\frac{d^2x}{dt^2} + kx = 0 \quad (1)$$

Free vibration with damping:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \quad (2)$$

[Piersol and Paez, Harris' shock and vibration handbook]



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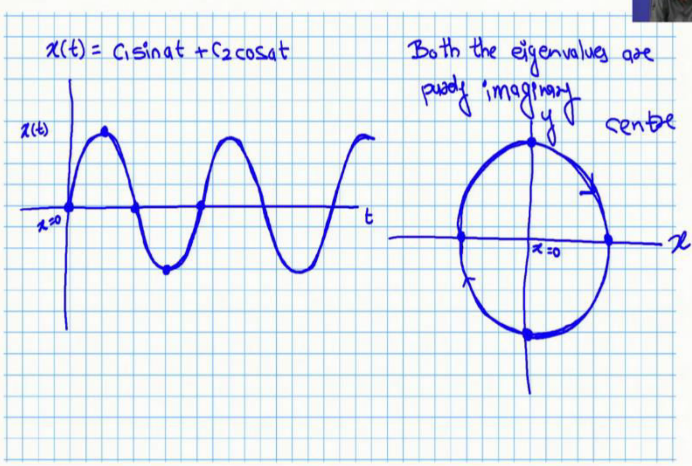

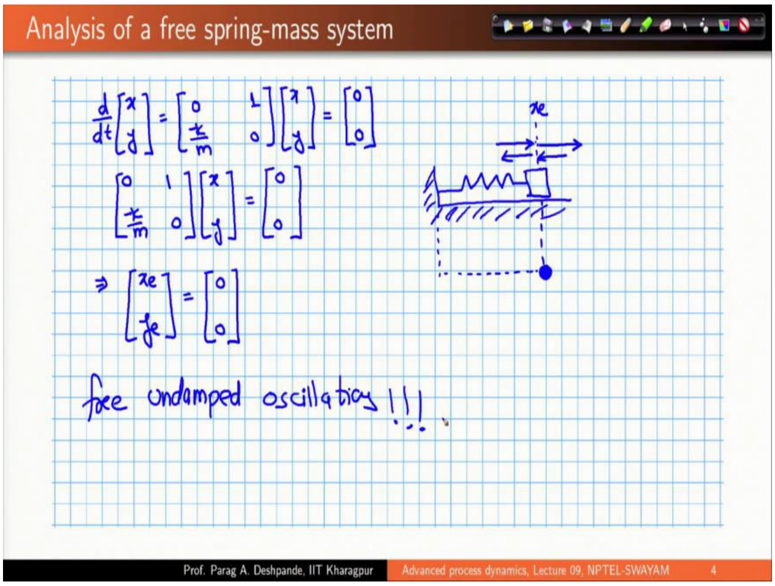


Hello. So, we continue our discussion on linear autonomous second order systems. In the previous lecture, we had a look into various phase portraits which are possible under different conditions, all of which satisfied similar equation which is given $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$ where $\underline{x}$ is the dynamical vector and $\underline{\underline{A}}$ is the it is the matrix. Today, we will take an example of a physical system.

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So, the example that we have today is a spring mass system. Now, whenever a spring comes into our mind, what do we think? We think about something which oscillates. So, we ask ourselves a question, can we have a spring which is not at equilibrium and still tries to come to equilibrium without oscillation. So, let me repeat if I give a disturbance to a spring, I expect it to oscillate.

Now, the question is, can I have a spring, which when given a displacement and its mass would like to come back would have a tendency to come back to the equilibrium but without oscillations. But, in fact, this happens. So, imagine that you have a car where you have the brake pedal, the clutch pedal and the accelerator pedal in all three pedals, you have springs, but what you do not observe is that as you retract any one of these pedals you do not observe any oscillations in them.

Similarly, when you have a door closer and you open the door and you leave it, the door tends to close and it happens without oscillations. So, therefore, you in fact can have a spring which has a tendency to come back to equilibrium position without oscillations. And now, we would like to ask, can we explain these different situations? The first situation being that the spring, when given a displacement in its mass keeps oscillating, then a second situation where it does not oscillate.

And in fact, there can be a third situation, for example, in a lot of bourdon tubes from measurement of pressure, you would have observed that the gauges fitted with oil. And that is because the system has a tendency to oscillate, to vibrate and to suppress those vibrations, oil is put. So, it is a viscous damping effect, which we want to put in there.

And in all these three cases, the governing equation remains the same, it is a second order ODE, which can be converted to a set of two first order ODE's and therefore, we have a system which is second order. So, now what we want to know or understand in this lecture is what are the different conditions under which the system will behave differently and what are all the possibilities which the system can sample.

So, we have here a spring mass system, so this is the spring and a spring is characterized by the spring constant. So, we have the linear spring which is characterized by the spring constant k and it is fitted with a mass m such that the motion is confined only along the direction of the axis of the spring and the system is assumed to have no friction.

So, there is no friction here, when the mass moves on the fixed surface and the governing dynamics can be obtained using simple Newtonian balance, which is a second law of, Newton's second law force balance and this is the balance equation $m\frac{d^2x}{dt^2} + kx = 0$ and we call such a system as free undamped system. Well, I will let you know what is the meaning of undamped, it will become clear a little later.

But the meaning of free is that if I give a displacement to the mass in one particular direction, and I leave the system to equilibrate, then after leaving the system to equilibrate, I am not providing the system with any extra external force. So, that is why it is called free. I am not supplying the system with any external force, except for giving a displacement to the system, which takes the system away from equilibrium.

So, this is a free undammed system. And then you as I made a mention for the case of bourdon tube, you can have viscous damping, you can put additional force on the system to make it damped.

So, what is the meaning of damping? Let us now define what damping is, well, you have a spring and it is supposed to we expect it to oscillate, we expect vibrations in the system. So, there would be oscillatory behaviour in the system. And if the system does oscillate with a constant amplitude with at a constant frequency, then you say that the system is undamped. So, you have certain amplitude here, let us say this is a and in the second cycle, again, if it is a and so on, then you say that it is undamped system.

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On the other hand, you can have a we will have a case where you give the displacement and the oscillations take place such that subsequent amplitudes $a_1$ $a_2$ are such that $a_2 < a_1$ and so on, the amplitudes of oscillations keep on reducing with time. Such cases are called free vibration with damping, free vibration because you are not providing the system with any external force. So, let us see if we can analyse the dynamics of the system.

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What we want to know is this. If we can convert the dynamical equations into matrix equations, we have a second order dynamical equation, and we learned that it is possible to convert them to first order linear ODE's. We will see whether that is possible in this case also. And once we do that, what are the equilibrium solutions? Can we develop the phase portraits and comment upon the stability and the effect of various parameters?

So, let us see what is the governing equation for in fact, for the two cases, we have

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \dots (1)$$

This is my first equation and let me convert it to a matrix equation for that, I will assume $\frac{dx}{dt} = y$. If this is the case, then my equation becomes

$$m\frac{dy}{dt} + kx = 0 \qquad \dots (2)$$

So, what I can do is I can rewrite equation 1 as the

$$\frac{dx}{dt} = 0x + 1y$$

And I can write the second equation, equation 2 as

$$\frac{dy}{dt} = -\frac{kx}{m} + 0y$$

So, I have converted my second order equation to two first order equations and how will I convert it to matrix equation. I can write,

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

So, I have converted my equation to matrix equation. Let us see if I can do this for the second equation.

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So, my second equation was

$$m\frac{d^2x}{dt^2} + C\frac{dx}{dt} + kx = 0$$

I will assume $\frac{dx}{dt} = y$; which means I have

$$m\frac{dy}{dt} + Cy + kx = 0$$

So, I can write equation 1 as

$$\frac{dx}{dt} = 0x + 1y$$

And

$$\frac{dy}{dt} = -\frac{k}{m}x - \frac{C}{m}y$$

which will put me in a position to convert this set of equations to a single matrix equation, which is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

So, now if I compare this equation and contrast it with the previous equation, which is

$$\frac{d}{dx}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

then I will see that the functional form of the two equations is identical. The equations are of the form $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$ but, the two matrices in the two cases are different. And we saw that yesterday that we can analyse the matrix and correspondingly depending upon the eigen vectors and eigen values of the matrices will get different behaviours. So, this is what we want to do today. So, let us see, let us consider the first case.

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So, my equation is $m\frac{d^2x}{dt^2} + kx = 0$ and I have converted this equation to

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

So, now what I need to do is I need to determine the eigen values of the system and corresponding eigen vectors. So, I have the eigen values and eigen vectors ready with me.

So, let me just write them down I have $\lambda_1$ which is the first eigenvalue this is equal to $-i\sqrt{\frac{k}{m}}$ and the corresponding eigen vector is $\begin{bmatrix} i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}$ and my second eigenvalue is obviously $i\sqrt{\frac{k}{m}}$ and the corresponding eigen vector is $\begin{bmatrix} -i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}$. I leave this as an exercise for you to make sure that this is what you also get by solving it by hand or by any other method.

So, if these are the eigenvalues and eigenvectors, then what would be our solution? Yesterday's lecture we know that the solution would be

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1\, e^{-i\sqrt{\frac{K}{m}}t}\begin{bmatrix} i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix} + C_2\ e^{i\sqrt{\frac{K}{m}}t}\begin{bmatrix} -i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}$$

and since x is the displacement and y is $\frac{dx}{dt}$ which is a gradient of the displacement or derivative, I am just at this point of time interested in knowing the dynamics of the system.

So, I am just interested in knowing the evolution of x with time. So, from this equation I can write

$$x(t) = C_1\, e^{-i\sqrt{\frac{K}{m}}t}\left(i\sqrt{\frac{m}{k}}\right) + C_2\, e^{i\sqrt{\frac{K}{m}}t}\left(-i\sqrt{\frac{m}{k}}\right)$$

this is my solution. This is a solution but this is not really very elegant and it does not tell me anything physical about the system. So, let me do one thing let me try to make the solution a little more elegant. So, what I will do is, I will convert i’s to cosines and sines.

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So, what I can do is this. I can write x of t is equal to C1 now I will convert $e^{-i\sqrt{\frac{K}{m}}t}$ to sines and cosines and this will become $C_1(cos\sqrt{\frac{K}{\dot{m}}}t + isin\sqrt{\frac{K}{\dot{m}}}t)i\sqrt{\frac{m}{k}}$ multiplied by i root over m by k plus $C_2\, e^{i\sqrt{\frac{K}{m}}t}$. So, there is one correction this will be minus, this will be minus.

So, this term would be $cos\sqrt{\frac{K}{\dot{m}}}t + isin\sqrt{\frac{K}{\dot{m}}}t$ everything multiplied by $-i\sqrt{\frac{m}{k}}$. So, this is what has converted my exponential of i took sines and cosines, still the solution is not very elegant. So, I will do further rearrangement what I will do is I will take this part and this part together with corresponding multiplication here. in fact, I will take yes, this part and this part and then I will take this part and this part together.

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So, when you take the real part and imaginary part together then what you would get is x(t) which is imaginary part times i plus real part. And then we know that imaginary part is a solution. And the real part itself again is a solution. So, imaginary part and real part together are solutions. So, when you would work that out, what you would get is that x(t) can be written as a linear combination of the imaginary part.

So, let us call it as $C_1 sin\ at + C_2\ cos\ at$. So, this will be the solution of your dynamical system. Now, $C_1$ $C_2$ and *a* would be expressed in terms of the parameters of the system. I am not here really worried about the exact expression, because I want to know the qualitative dynamics of the system and what I see from this functional form is that x(t) would vary as $C_1$ some multiplicative constant which would be in terms of the parameters of the system, times *sin at*, *a* again would be in terms of the parameters of the system plus $C_2$, a multiplicative constant times another function which is cos and that again would be *cos at* would be again be the function of the parameters of the system.

So, now what we want to know is that if this is the functional form how would the system behave? So, let us look into that.

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So, let me punch in the values here $f(x) = C_1\ sin\ kx + C_2\ cos\ ax$. I see this behaviour, but let me do a further diagnosis this itself is very clear, but let me first make $C_2 = 0$. So, when $C_2$ is 0 you have a pure sinusoidal behaviour and you can see this particular behaviour to continue in time pure sin.

Then I will make $C_1$ some non-zero value and $C_1$ as a 0 value and $C_2$ some non-zero value and you see some you see a pure cosine curve and when you have $C_1$ and $C_2$ both you have you have an effect from both sin as well as cosine and what you see is that it does not matter what $C_1$ and $C_2$ are, but the characteristic feature of this system is that you have oscillations which are sustained, first feature is that the oscillations are sustained. you do not see any behaviour where this oscillations do not sustain.

Secondly, what about the magnitude of oscillations. You will see here that in all the cases the magnitude of assertions for example, the magnitude here is 2.23, here also it is 2.23, here also it is 2.23 and so on. So, as you go further in time, irrespective of what time has been reached, you always see the same amplitude of the system.

And that is why we call the system as if you notice here, undamped system, so, the system is called undamped because there is no damping. So, if I have vibration or oscillation of certain magnitude then in time this magnitude is going to continue in the same amplitude. So, therefore, this is there is no damping in the system first point there is no damping and second point is that there is no seizure of the oscillations at all oscillations continue.

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So, therefore what we saw here is that you have solutions of the form x(t) which is equal to $C_1 \sin at + C_2 \cos at$ and the solutions look like this. They have same sustained amplitude, but now, in this the x axis is t, the y axis is $x(t)$. And if I want to know what would be the corresponding phase portrait for such a system, then what will I get?

Well, I found out that both the eigen values are purely imaginary. And what kind of phase portrait do I make when both the eigenvalues are purely imaginary? Well, I can do this, I can make here x for all the current case it would be y. And if I remember what I learned yesterday, the face portrait would be a centre face portrait.

Now, here you have x versus y in the face portrait in the solution, you have t versus x. So, is there a correspondence between the two? Or do they match. And it is not very difficult to see, as I said, what is going to happen is that you will circle around these values in time. So, therefore, what is going to happen is that the arrow of time is going to be made something like this. So, imagine that I start from this value here, at which x is equal to 0.

And this corresponds to this, $x = 0$, now my value of x is increasing, and it reaches a maxima. With time, this is the arrow of time, that is what I see here also, I go and reach a maxima, once I reach a maxima, the value of x starts decreasing, it reaches a minimum value, well, that happens here. And then, what happens is it in fact, it reaches the value 0 not minimum value, it reaches the value 0, and then my time continues here.

Now, I have negative x here, in fact, I am going to negative x and I am going to the minimum value this is the minimum value of x and then I continue here I again go to x = 0. And once I reach this, when I complete one circle in the phase portrait, I complete one oscillation and then when I start the second circle, I keep on repeating the same values the same values keep on repeating.

And therefore, here also I keep on repeating the same values without changing the amplitude or any of the values in the subsequent cycle and therefore, the oscillations would continue indefinitely forever and in the phase portrait about the centre, your circle, the point will keep on and circling around and giving you the same value over and again. Now, the only point which is left for us to discuss in this particular problem is that what is the equilibrium solution? We did not worry about equilibrium solution till now.

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So, we had the equation

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

And what is going to happen? When the system reaches equilibrium, the equilibrium solution will be given by the 0 vector, this is important, since I have a dynamical vector, the equilibrium solution would be a 0 vector, which means that all the components of the vector would be 0.

So, I simply need to solve

$$\begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which very simply gives me x equilibrium, y equilibrium as 0, 0. And again, does it physically make any meaning? Yes, well it does, what is the equilibrium position if you have a spring, So, I have a spring and I have a mass and I give a small, so this is the, let us say this is the equilibrium position x equilibrium.

So, I give a positive displacement to the system, the system has a tendency to come back here. I give a negative displacement which means I compress the spring, the system has a tendency to come back to again to the equilibrium position, so equilibrium position here means that the difference that the displacement define with respect to this particular point which you call as the equilibrium point is simply 0, you define that point as the equilibrium point, or you also call it as unstretched so, the co-ordinate corresponding to the unstretched length of the spring.

So, this was the case when you have a undamped system, so undamped free, undamped free oscillations. What were the features, the magnitude or magnitude of oscillations do not change with time, they sustain. In the next lecture we will see that it can happen that with time the magnitude of oscillations may change or there may be no oscillations at all, under what conditions will it happen, we will take this up in the next lecture, thank you.