Advanced Process Dynamics
Professor Parag A. Deshpande
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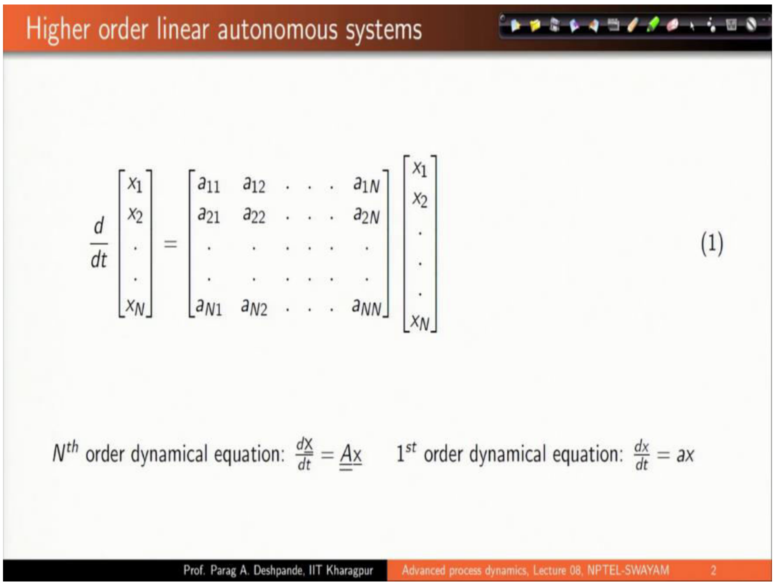
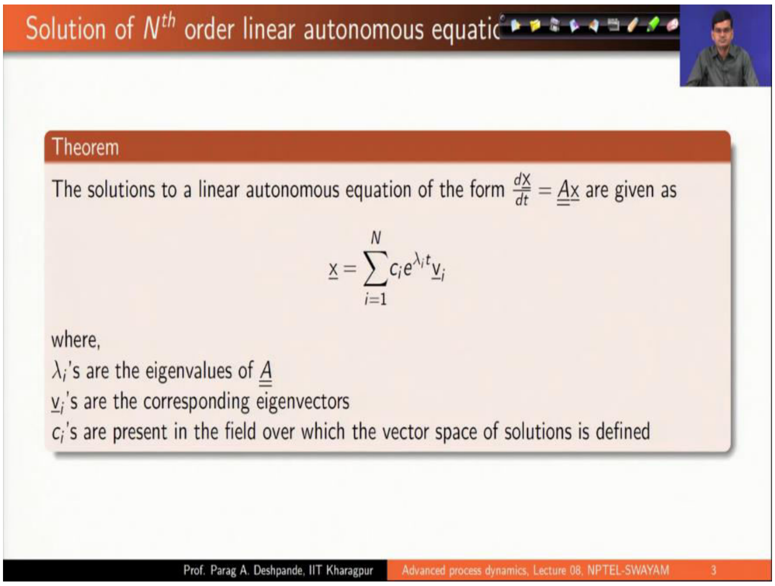
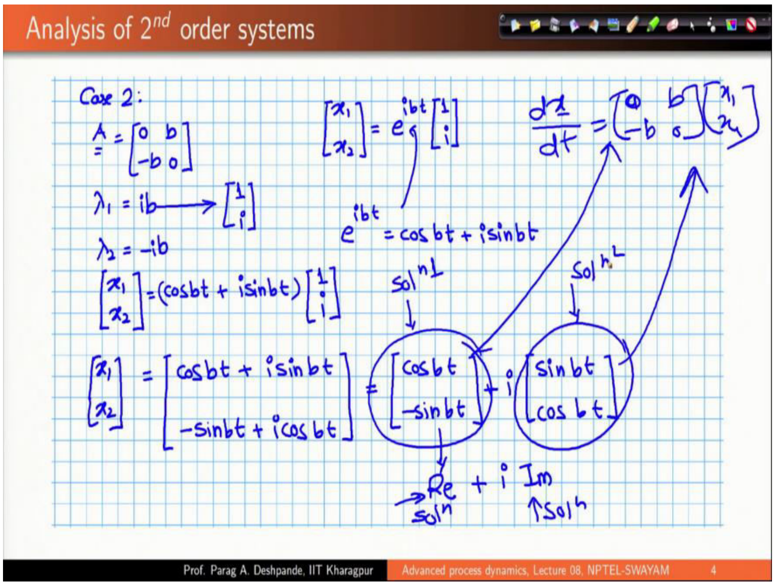
Lecture 08
Phase plane analysis of linear autonomous second order systems continued






Analysis of 2nd order systems
Case 2:
A = [0 b; -b 0]
λ1 = ib → [1; i]
λ2 = -ib
[x1; x2] = (cosbt + isinbt) [1; i]
[x1; x2] = [cosbt + isinbt; -sinbt + icosbt] = [cosbt; -sinbt] + i [sinbt; cosbt]
Re + i Im
soln
soln
[x1; x2] = e^{ibt} [1; i]
e^{ibt} = cos bt + isinbt
dx/dt = [0 b; -b 0] [x1; x2]
soln 1
soln 2
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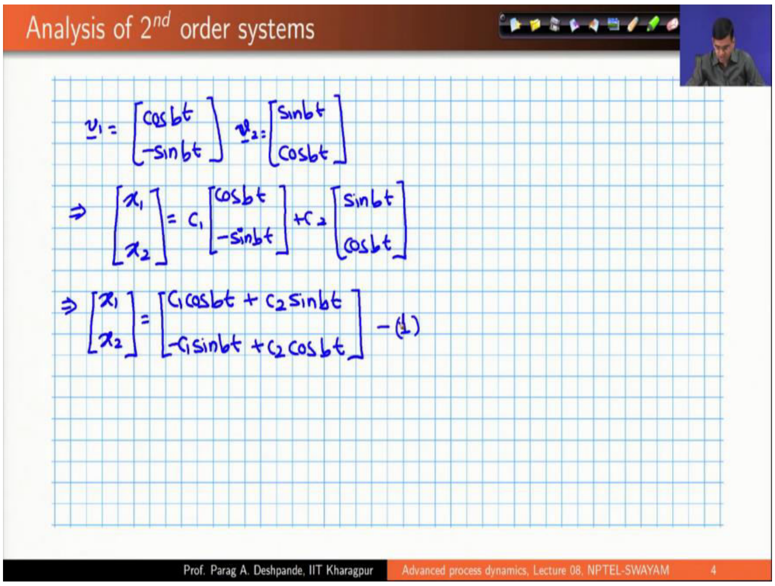
4


Analysis of 2nd order systems
v1 = [cosbt; -sinbt] v2 = [sinbt; cosbt]
⇒ [x1; x2] = c1 [cosbt; -sinbt] + c2 [sinbt; cosbt]
⇒ [x1; x2] = [c1cosbt + c2sinbt; -c1sinbt + c2cosbt] — (1)
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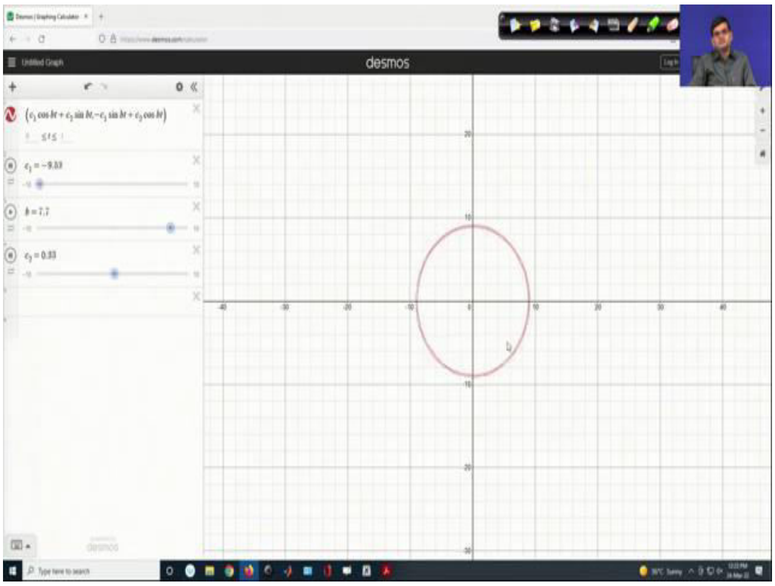
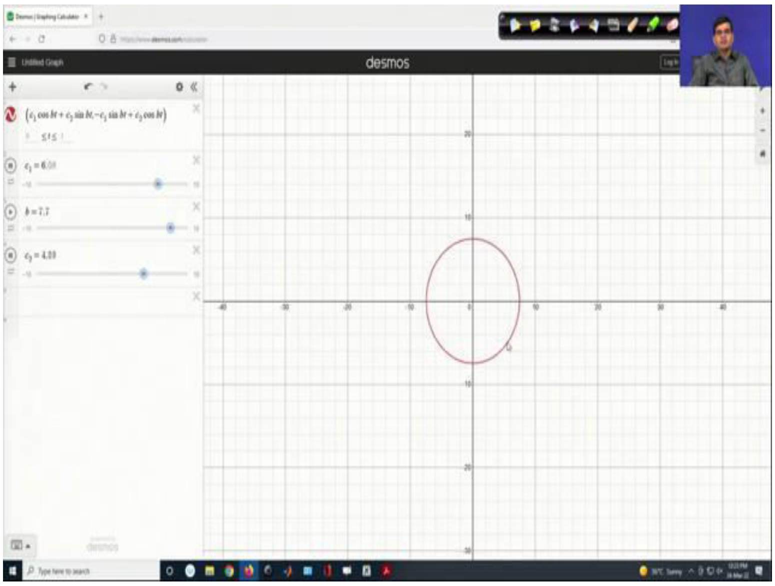
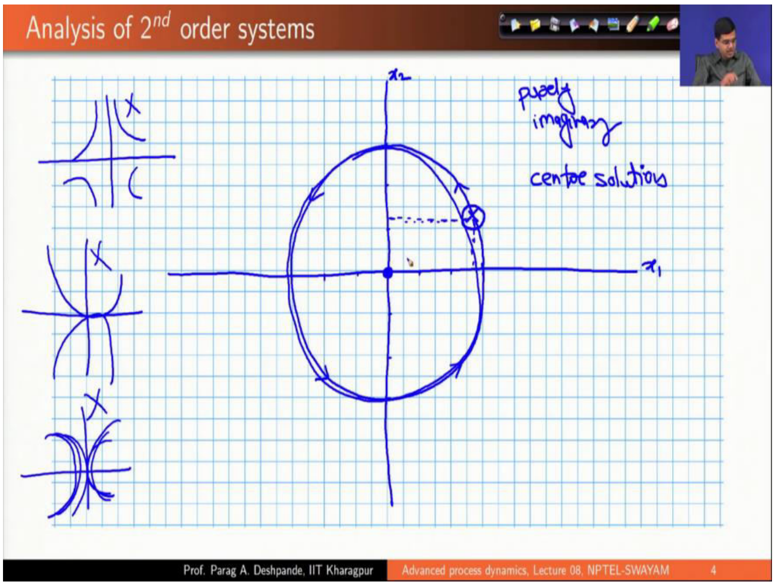

desmos


Untitled Graph
desmos


Analysis of 2nd order systems
purely imaginary
centre solution
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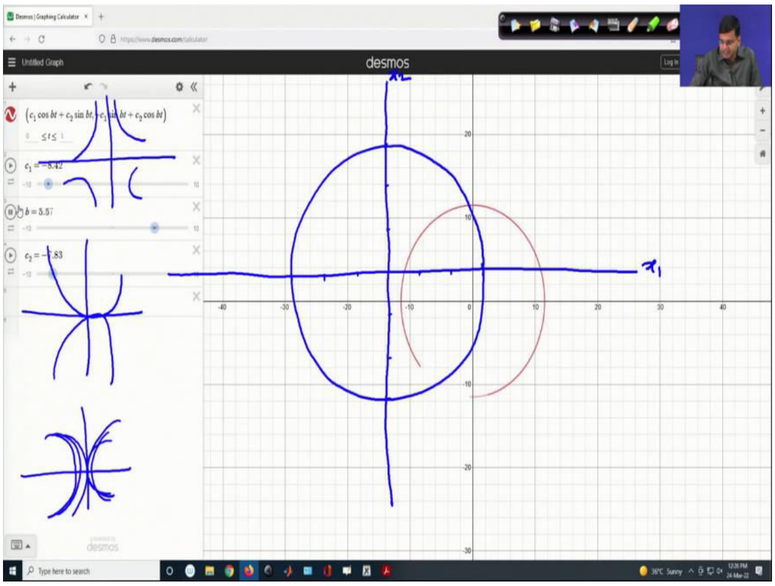
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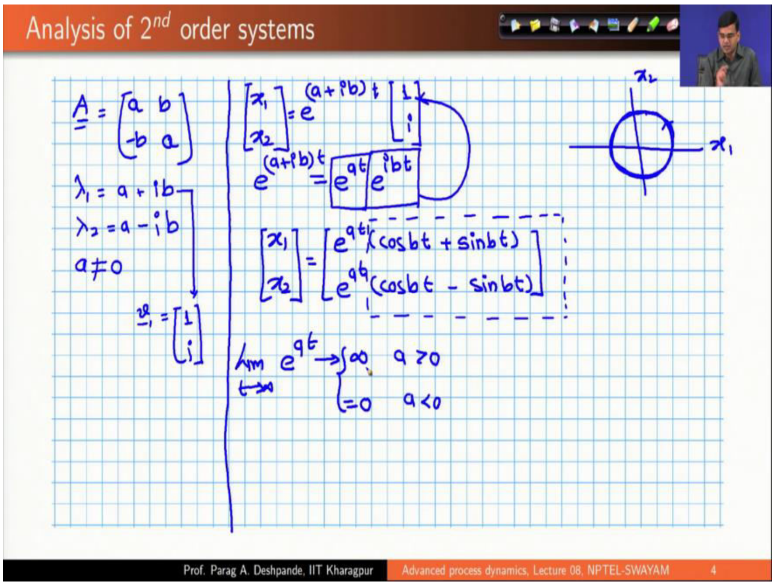
Analysis of 2nd order systems
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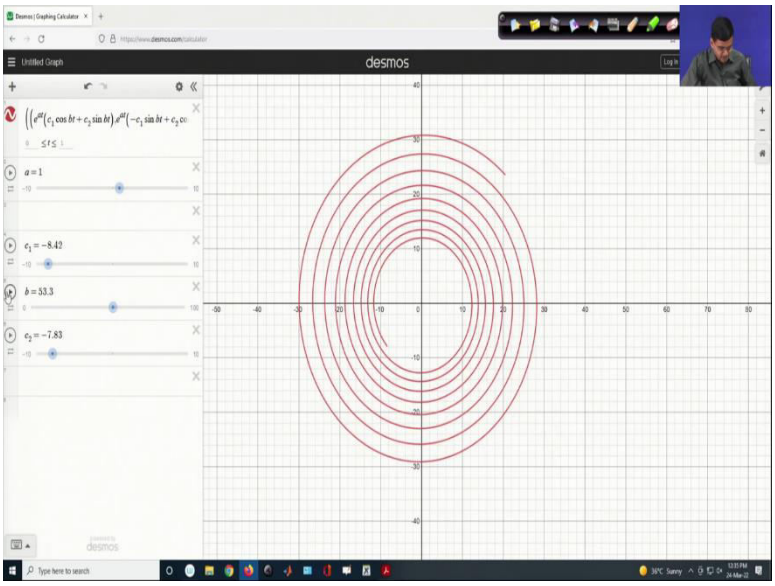
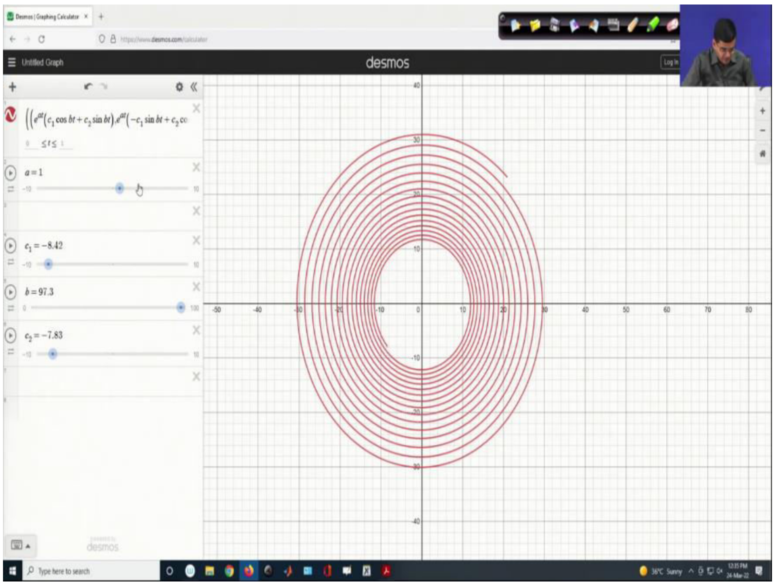
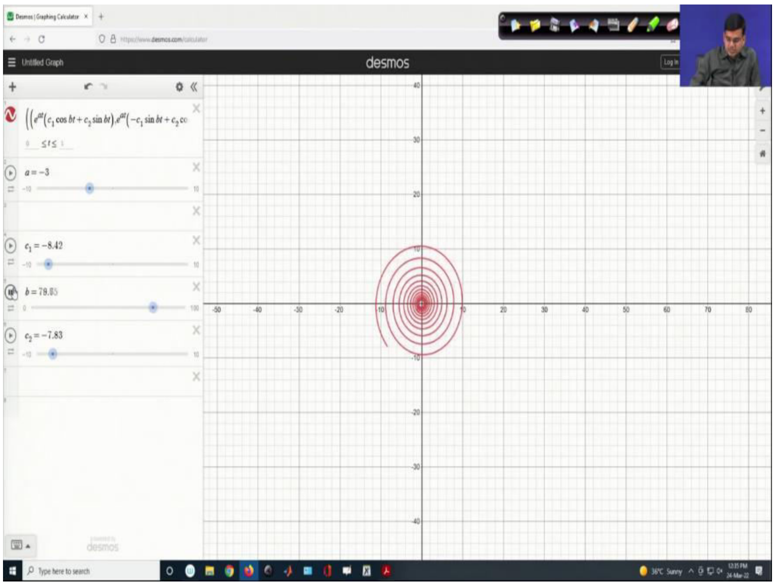
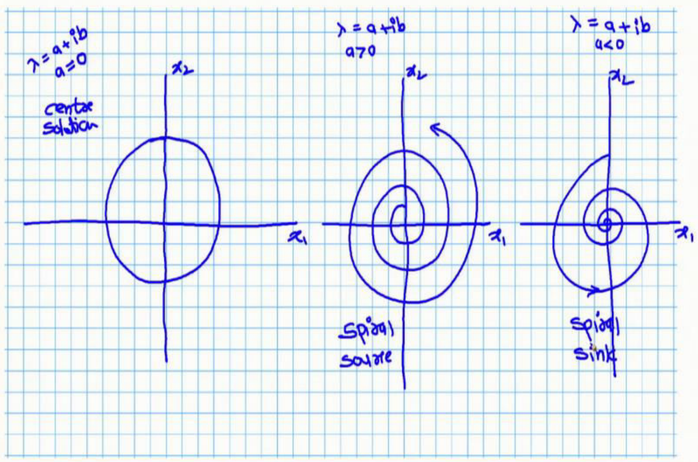

desmos
a = 1
c1 = −8.42
b = 53.3
c2 = −7.83

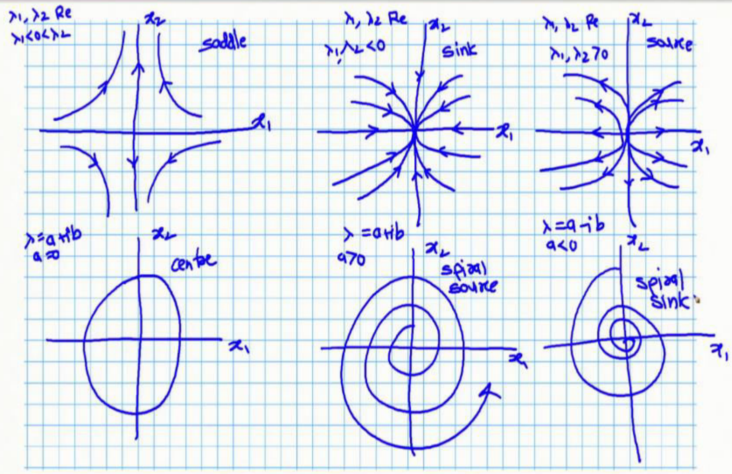

desmos
a = 1
c1 = −8.42
b = 97.3
c2 = −7.83



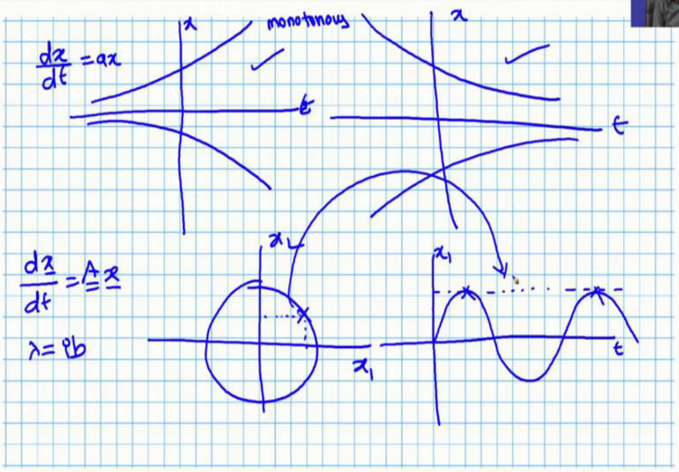
Analysis of 2^{nd} order systems



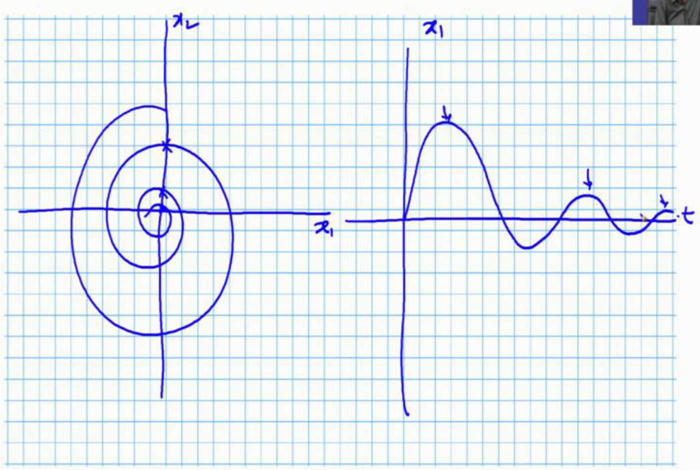
Analysis of 2^{nd} order systems



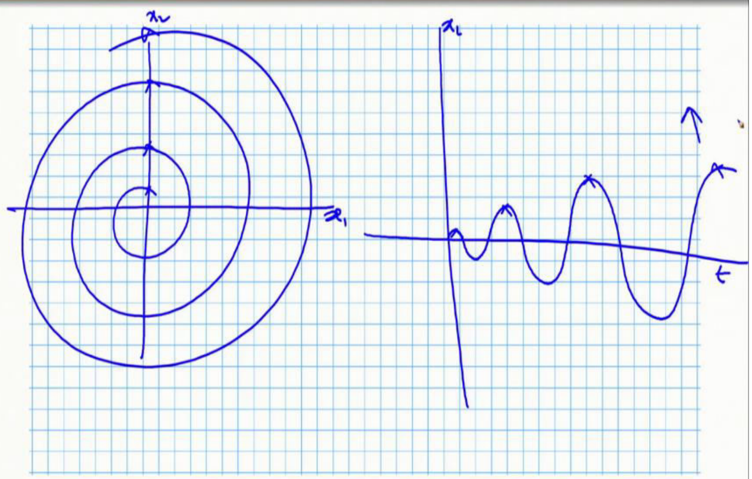
Analysis of 2^{nd} order systems



Analysis of 2^{nd} order systems



Analysis of 2^{nd} order systems



So, let us continue our discussion on the phase plane analysis of second order systems. In the previous lecture, we took the example where the eigen values were real and we took three cases, where we had λ_1 and λ_2 both positive λ_1 and λ_2 both negative and in the third case, one of them was positive, the other one was negative.

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So, as a quick reminder, we are looking at higher order systems which can be expressed in this particular form and the solution was given as

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

where λ_i are the eigen values and V_i's are the corresponding eigen vectors.

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Now, let us look into a particular form of the matrix A. So, we have case 2 no, okay. So, we have $\underline{\underline{A}}$ which is given as $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$, the eigen values which we considered in the previous lecture were all positive were all real numbers, they were not imaginary numbers, but if you solve for eigenvalues and eigenvectors for this particular case, what you will find is that λ_1 is going to be *ib* and λ_2 is going to be *-ib.* They are going to be imaginary numbers.

And the corresponding eigen vector so, let us take one of them, the eigen vector for e^{ib} can be determined as $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and therefore, what you can do is that you can write the solution as,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{ibt} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

The solutions would be linear combinations of both the terms we are considering for the time being only one of the terms to understand the behaviour of the system.

So, this is the solution and what I will then do is, I will convert e^{ibt} like this $e^{ibt} = \cos bt + i \sin bt$. So, I get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\cos bt + i \sin bt)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

which is going to be

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos bt \ + \ i \sin bt \\ i\cos bt \ - \ \sin bt \end{bmatrix}$$

I can further simplify this as,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cos\ bt \\ -\ sin\ bt \end{bmatrix} + i \begin{bmatrix} sin\ bt \\ cos\ bt \end{bmatrix}$$

So, what you see here is that I have solutions which are complex numbers. So, I can write this as real part of the solution plus imaginary part of the solution multiplied by i and it is not very difficult to show this by simple substitution that the real part is a solution and the imaginary part is a solution individually.

So, if you substitute this real part in

$$\frac{dx}{dt} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

then you will see that you will get this part satisfies this equation and this part also satisfies the equation. So, if that be the case, I have solution 1 and I have solution 2. So, let me do one thing let me write this here solution 1 as $\begin{bmatrix} cos\ bt \\ -\ sin\ bt \end{bmatrix}$ and $\begin{bmatrix} sin\ bt \\ cos\ bt \end{bmatrix}$.

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So, cos bt minus sin bt and sin bt cos bt. So, v_2 is a solution and v_1 is also a solution if v_1 and v_2 are individually solutions of my linear system then I know that their linear combination would be also a solution. So, I can write the ultimate solution of my system as $C_1 \begin{bmatrix} cos\ bt \\ -\ sin\ bt \end{bmatrix}$ $+C_2 \begin{bmatrix} sin\ bt \\ cos\ bt \end{bmatrix}$, let me reiterate that the reason we are writing this expression is that the system is linear it follows the property of linearity and therefore, all linear combinations of the solutions would also be solution.

So, therefore, my solution is of this form from where I can write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} cos\ bt \\ -\ sin\ bt \end{bmatrix} + C_2 \begin{bmatrix} sin\ bt \\ cos\ bt \end{bmatrix}$$

and now, I would be interested in knowing the behaviour of x_1 and x_2 when they are given by this equation. It is a little difficult to visualize this particular parametric plot so, let us do one thing let us plot it using our Desmos calculator.

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So, we will write this like this. So, I have

C_1 cos bt + C_2 sin bt, - C_1 sin bt + C_2 cos bt. So, before I do some rearrangements let me clear this up. So, now what I can do is, I can increase the value of b and you see what you get, you get a circle as the solution, let me change the values of C_1 and C_2 and let me animate it, what I see is that when I change the values of these multipliers C_1 and C_2. The radius of the circle changes, but what does not change is the nature of the solution, what does this mean?

This means that the solution has the same circular behaviour irrespective of the values of C_1 and C_2. So, what do I learn from this how do I draw the phase portrait how do I draw the phase portrait?

So, as a saw from the Desmos calculator my x_1 and x_2 plane would look something like this, I am trying to draw it like a circle. It is not very elegant, but this is what I wanted to draw. Probably we can do something better in this part. Yeah, it is better now. So, now what I see is that I go along a circle, remember that previously you had one solution which looks, which looked like this in one case the values were real in other case the solutions were like this and in one more case the solutions for like this and so on, but the value is never repeated what is the meaning of this? What is the direction of time now?

Now, the question is what is the direction of time? You have if I animate this with respect to b, you will realize that now, you see the direction of time you can see the circle evolving in time. So, depending upon the parameter b in your case you either have clockwise or anti clockwise evolution in time, but, in any case, once the system has completed a circle, once the system has completed a circle, the system keeps on keeps on revolving either in counter-clockwise direction or clockwise direction.

So, that means, that the value so, for example, if I am here at some point of time t, so, I will have some value of x_1 and some value of x_2 when the system completes this circle which means, after one period of this circulation is over my value will again be the same that means, the system has repetitions, this repetition was not observed here, this repetition was never observed here, this repetition was never observed here.

But when the eigen values are purely imaginary this is important purely imaginary not just complex numbers, when the system when the eigenvalues are purely imaginary, then what you see is that you have a dynamical behaviour which looks like this and therefore, such solutions are called centre solutions. Because about a centre the solutions keep evolving. And this happens this is a typical behaviour when you have a system where the eigen values are given as purely imaginary numbers.

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Now, imagine that you have a system where $\underline{\underline{A}}$ which was initially given us $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ is given now as $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. So, the diagonal elements which were previously 0 are now a, in this case the eigen values become $a + ib$ and λ_2 becomes $a - ib$. So, in the previous case you had the eigen values, which were purely imaginary. Now you have complex numbers, a cannot be 0.

So, let me emphasize that $a \neq 0$. So, what is going to happen to the solution, the solution would be something like this, let us consider the first case the eigen vector is the same, which is 1i. Please, work on yourself to convince yourself that this is the eigen vector that you would get.

So, what you are going to get is this

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{(a+ib)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

So, I will write this as $e^{(a+ib)} = e^{at}e^{ibt}$. So, now what has happened is that your solution has now been multiplied by $e^{at}e^{ibt}$ remains the same right and then you will multiply here and carry on with the exact same steps.

So, therefore what is going to happen is that my ultimate solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is going to look like this e^{at} (cos bt + sin bt), e^{at} (cos bt - sin bt).

So, this is the plus sin bt and - sin bt, and what has happened in this case is that the entire term has been simply multiplied by e^{at}. But what is the effect of multiplication of e^{at} to this circular behaviour. If I consider only this behaviour, which was the previous solution, if I consider only this behaviour, the previous solution my behaviour was like this x_1, x2 and I have a centre solution depending upon the value of b you would have one or the other direction of the arrow. But now, I have multiplied this effect, this circular effect by e^{at} .

Now, what is the behaviour of e^{at}, we know that e^{at} limit t tends to infinity, tends to infinity for $a > 0$ and it is equal to 0 for $a < 0$. We know this functional form which means that this circular effect, the effect of keeping the radius of the circle constant will be augmented by the term and the term will keep on increasing with time for $a > 0$ and the term will keep on decreasing with time diminishing to 0 for $a < 0$.

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Simple so let us see if this is the case let us put this here. All we need to do is we need to multiply so, what we need to do is we need to multiply this by e^{at}, yes and here also e^{at}, let us see what happens here. But clearly without doing anything you saw that there is no more a circle of constant radius but what has the software done is that, it has assumed the value of a is equal to 1 which means $a > 0$. So, that means you started from here and instead of reaching a constant value you, so, this would have been your constant radius circle roughly you started diverging, started diverging.

And let me increase the time. So, I can increase the time by doing something like this. So, let me do it. See here so, I can do one thing from 0 to 100 and when I increase the time remember that my value of $a > 0$ and what I see is that I keep evolving.

So, let me see, let me show that the system in fact diverges from the initial value to a larger value, what happens when the value of $a < 0$? You see here. So, let me again make the system evolve with time, this is how the system is evolving with time it is tending to 0 asymptotically. So, as I increase the time, the value of my x and y asymptotically tend towards 0.

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So, therefore, now I have three types of phase portraits, x_1 x_2, x_1 x_2, x_1 x_2. The first portrait was this when λ was a + ib with $a = 0$, this is the centre solution. Now, I have λ is equal to a + ib with a greater than 0. This is important a greater than 0. So, I start with some point and then as time increases I tend towards infinity.

So, what should I call such solutions? These are called spiral. Why spiral, because that is spiral, spiral source and in contrast when λ is a + ib, when $a < 0$, you start with a point and then you spiral down towards 0 and this is spiral. This would not be spiral, but spiral sink. So, let us very quickly draw all the possibilities which we saw today.

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x_1 x_2, x_1 x_2, now case 1, when λ_1 and λ_2 are real, and $\lambda_1 < 0$, $\lambda_2 > 0$. So, what you get is a saddle solution, since $\lambda_2 > 0$, you would have the second axis as the divergent axis and therefore, this would be the nature of your solution. Second case λ_1 λ_2 are real and λ_1 and λ_2 are both less than 0.

So, depending upon whether $\lambda_1 > 0$, $\lambda_2 > 0$, you will get one or the other directions of these curves, but in either case you would have the lines which would look like this and this would be a sink solution. The previous solution was source, the third one λ_1, λ_2 are real and both of them are greater than 0.

Again, I must emphasize that depending upon the, whether λ_1 is greater or λ_2 is greater, you will get one or the other orientations of these curves. But, the direction of time would be out, outward, both the axis would be unstable. Now, λ is given as $a + ib$, where a is 0. So, this is, okay there is one correction this is not source, this is a saddle one please note this and this is a source solution.

So, when you have λ is of the form $a + ib$ where a is 0 which means you have purely imaginary eigen values then you get a centre solution then λ is equal to $a + ib$ where $a > 0$ your solutions look like this and these are called spiral source and λ is equal to $a - ib$ where $a < 0$ you get spiral sink solution.

Now, finally, what we have not discussed or what we have not seen in these phase portraits is that what about the dependence of the variable or the components of the dynamical variable on time, because these plots give you the variation of one component of the dynamical variable with the other component of the dynamical variable. What if I want to know x_1 with time or x_2 with time and so on.

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And what is the most striking feature of second order and higher order systems is like this. That when I solve for dx by dt is equal to ax then what I see is that my solution would either be like this or it would be like this. This is t versus x, this is t versus x which means in either cases the solution is monotonous.

The solution would the values would monotonously increase with time as in the case here or they would decrease with time, sorry, the other way around, they will either increase with time as in the case here or they will decrease with time as in case here. But now, when you are solving $\frac{dx}{dt} = ax$ such that λ for a is simply of the form ib then what you see is a centre solution x_1 x_2 and the values will repeat with the same magnitude the values will repeat with the same magnitude over a long period of time. So, that means, I can draw roughly the plot of t versus x_1 for example, like this.

Because these values are the same, the magnitudes are the same. So, a centre solution would correspond to this non-monotonous behaviour, this behaviour is non-monotonous which you will never find in first order systems.

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And finally, when you have spiral source or spiral sink solutions, this is spiral sink for example, if I want to correspondingly draw t versus x_1, this is x_1 and x_2 at any given point of time, what is going to happen is that magnitude is decreasing. So, therefore I can say that I have a behaviour which is say that like this. Where the amplitude is decreasing.

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Conversely what you can also have for the spiral source is this that at any given time the amplitude is increasing, so therefore you may have a system which behaves like this, that the amplitude of your system is actually increasing with time. So, we saw how the eigen values of the system govern the dynamical behaviour of the system, whether the eigen values are positive or negative whether the eigen values are real or imaginary or whether the real part of the eigen values is positive or negative will determine the future behaviour or the dynamical behaviour of system, whether the system would be convergent or divergent, whether the system would change monotonically or it would oscillate whether the oscillation would be sustained oscillation or they would die out with time or they would increase in amplitude with time.

These are all exotic feature which you observe only in higher order system which are not observed in first order systems. So, in the next lecture we will take an example from the physical domain and see how the features that we develop today can actually be observed in real world, thank you.