Advanced Process Dynamics
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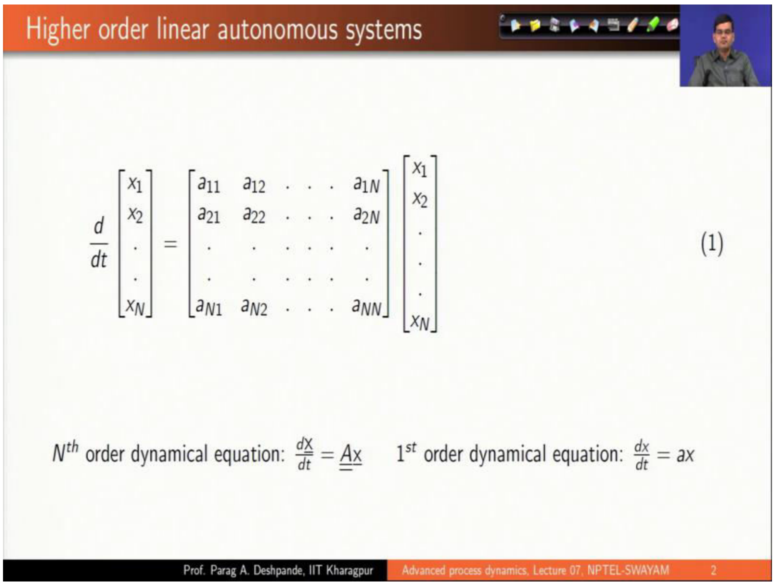
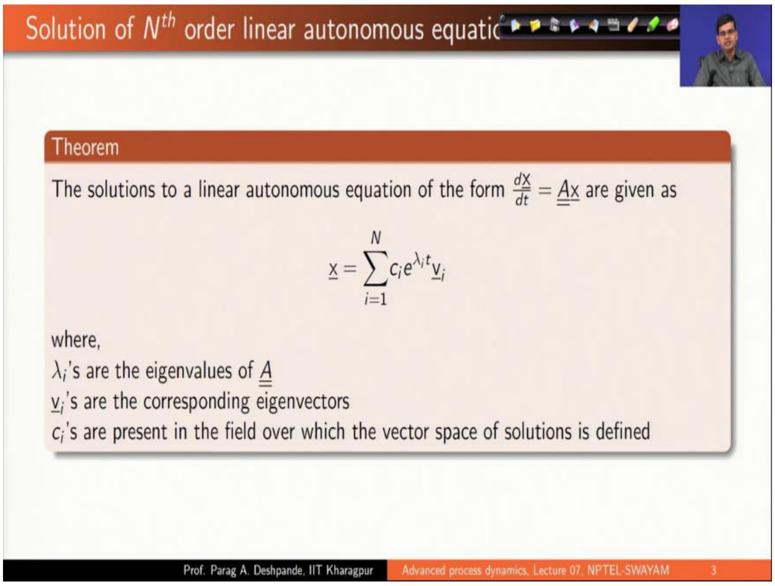
Lecture 07
Phase plane analysis of linear autonomous second order systems





$$\frac{dx_1}{dt} = ax_1 + bx_2$$
$$\frac{dx_2}{dt} = cx_1 + dx_2 \qquad -(1)$$

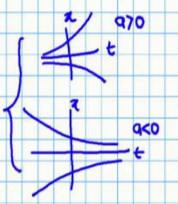
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad -(2)$$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$

$$\frac{dx}{dt} = ax$$

eigenvalue

$\frac{d}{dt}$



Case I: $\underline{\underline{A}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$\lambda_1 = a$; $\underline{v}_1 = [1 \;\; 0]^T$

$\lambda_2 = b$; $\underline{v}_2 = [0 \;\; 1]^T$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{at}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{bt}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad -(3)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{at}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{bt}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$a<0$; $b>0$

At $t=0$, and for $c_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $c_1 = 1$; $c_1 = 2$

$[1 \;\; 0]^T$ $\quad$ $[2 \;\; 0]^T$

At $t=0$ and for $c_1 = 0$

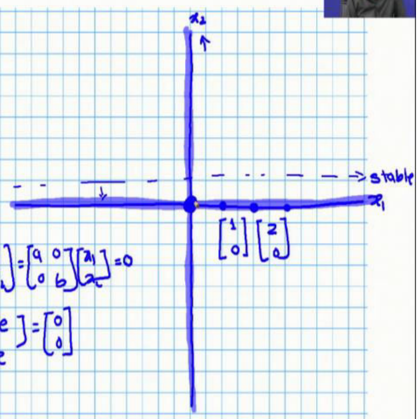
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Case I(a) $a<0$; $b>0$

$\lim_{t\to\infty} e^{at} = 0$ $\quad$ $\lim_{t\to\infty} e^{bt} \to \infty$

$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{at}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{bt}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$a<0$; $b>0$

At $t=0$, and for $c_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $c_1 = 1$; $c_1 = 2$

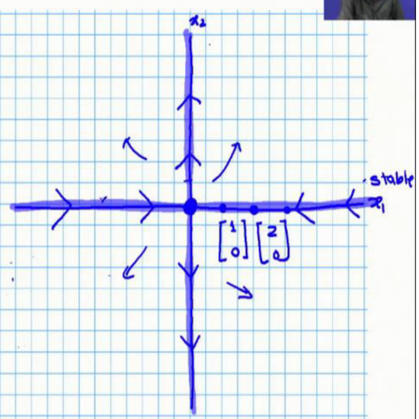
$[1 \;\; 0]^T$ $\quad$ $[2 \;\; 0]^T$

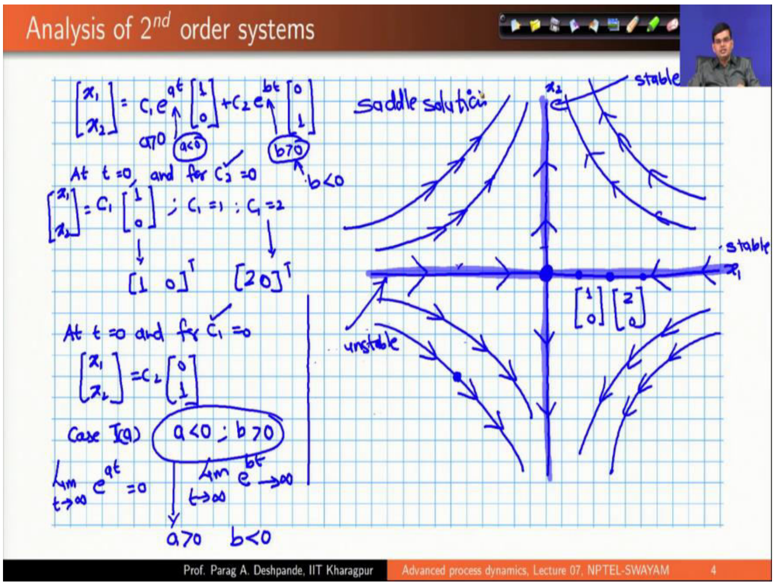
At $t=0$ and for $c_1 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Case I(a) $a<0$; $b>0$

$\lim_{t\to\infty} e^{at} = 0$ $\quad$ $\lim_{t\to\infty} e^{bt} \to \infty$




Analysis of 2nd order systems
saddle solution
stable
unstable
At t = 0 and for $c_2 = 0$
At t = 0 and for $c_1 = 0$
Case I(a) a < 0 ; b > 0
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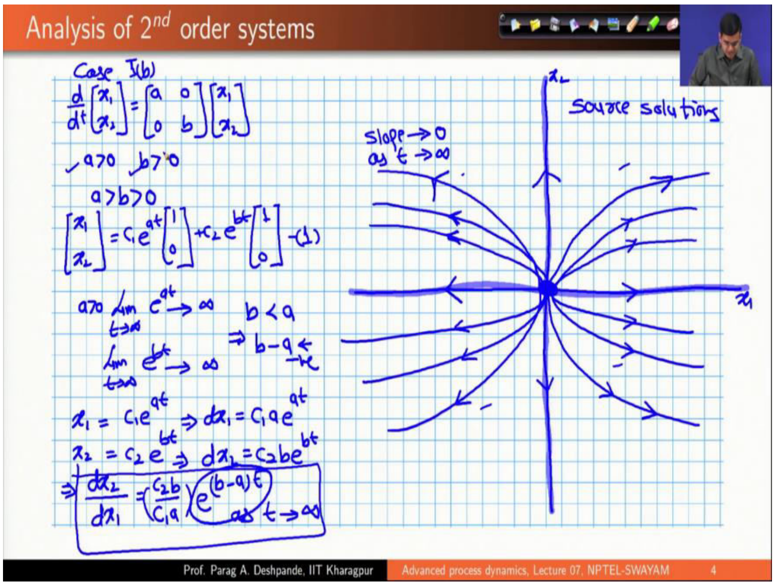
Analysis of 2nd order systems
Case I(b)
Source solutions
slope → 0 as t → ∞
a > 0 b > 0
a > b > 0
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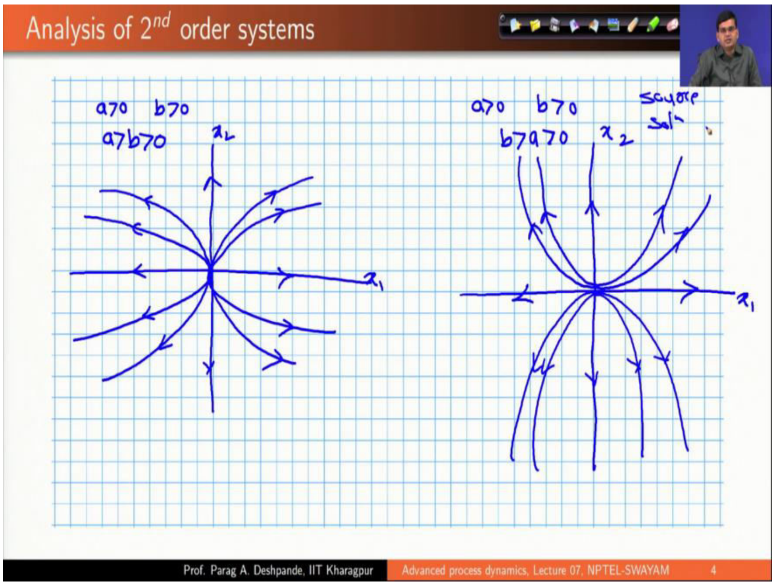
4


Analysis of 2nd order systems
a > 0 b > 0
a > b > 0
b > a > 0
Source soln
Prof. Parag A. Deshpande, IIT Kharagpur
Advanced process dynamics, Lecture 07, NPTEL-SWAYAM
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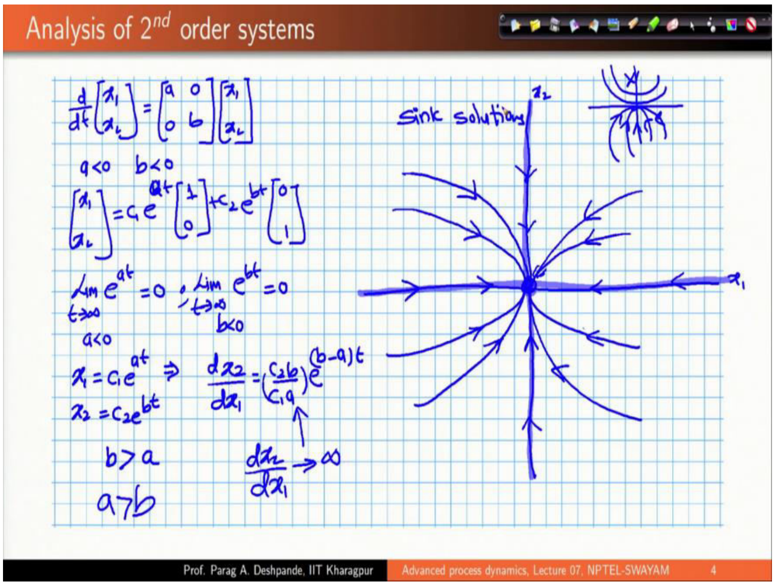

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So, we continue our discussion on higher order autonomous linear systems. In the previous lecture, we took a general n^{th} order system and we said that an nth order system can be represented by a matrix equation, which is given as d/dt of the dynamical vector which is multiplied by the matrix A multiplied by the vector itself. And we said that there is a very close similarity to the first order dynamical equation, where you can simply substitute the first order variable x by the vector $\underline{x}$ and the variable A by the matrix $\underline{\underline{A}}$.

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To solve such equations, we said that the solution is given by $\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$

where λ_i's are the Eigen values of the matrix and V_i's are the corresponding eigenvectors, we said that we will take the case of second order systems because of their relevance and therefore, we will dedicate this lecture specifically for understanding the behaviour, the dynamical behaviour of second order systems.

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So, let us look into typical second order systems. So, as I have mentioned previously, I can write an autonomous second order system as representative system as

$$\frac{dx_1}{dt} = ax_1 + bx_2$$

$$\frac{dx_2}{dt} = cx_1 + dx_2$$

Now, the first step to do this analysis would be to convert this system of equations to a matrix equation. So, I would write this set of equations as a single matrix equation as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This is my dynamical equation, let's remind ourselves that this is of the form $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$

When I wanted to look into the dynamical behaviour of the first order equation, which was $\frac{dx}{dt} = ax$, I basically saw that there are two behaviours one is exponentially growing behaviour and this was for a greater than 0 and the other was exponentially decaying behaviour and this was for *a* less than 0 and you would see that *a* in fact is an eigenvalue of the operator d/dt, a in the previous case is an eigen value of the operator d/dt where you can say that x is the eigen function.

So, similarly in our current case I can look into the eigen values of *a* and comment upon the behaviour in the previous case the system had a bifurcation at $a = 0$. So, the value of *a*, the sign of *a* determine the fate of the system, in case of second order systems what I would need to do is I would need to look into the eigen values not simply a but all the eigen values of the matrix A to determine the future of the system.

So, let us focus on one of such cases. So, I have case 1, where my matrix is of the form A. It is of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, suppose you have a dynamical system which can be modelled such that the matrix A is nothing but $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ in such a case, I encourage you to determine yourself that λ_1 is *a* and λ_2 is *b* and correspondingly V_1 is $[1\ 0]^T$ and V_2 is $[0\ 1]^T$.

So, now if this be the case then what would be my solution? My solution would be x1 x2 is equal to $C_1\, e^{a\,t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2\, e^{bt}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This is my solution, the solution to the system of equations of the form which is given by equation 1 or the corresponding matrix equation 2 would be given by equation 3.

Now, how would I determine the general behaviour or nature of the solution which is given by equation 3, I will need to look into the magnitudes as well as the signs of the eigen values. So, let us consider case 1a.

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So, I have so, my general solution is given by this

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1\, e^{a\,t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2\, e^{bt} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, if I want to plot the variation of x_1 and x_2, then how would I do? Let us say that I plot x_1 versus x_2 at time t = 0 and for $C_2 = 0$ because I will have all combinations of C_1 and C_2 and all of them would satisfy the solution. So, one of those would satisfy the condition where $C_2 = 0$. So, at t = 0 and C2 = 0 my solution is simply C_1 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and further I can set C_1 as 1 which would give me simply the solution as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So, therefore, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a solution, but now I can set C_1 as 2 which will give me $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a solution which means I have this $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a solution and I can go on and this will give me this entire line as the solution line. So, this entire axis is in fact a solution to my system of equations. Similarly, now I can do at t = 0 and for $C_1 = 0$, I can write x_1 x_2 as C_2 $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and following the exact same analysis, now I can write, I can draw this entire x_2 axis as the solution to my system of equations. fine.

Now, the question is what is the nature of the curves which go along x_1 and x_2. So, now I have a case. Case 1a, where I say that $a < 0$ and $b > 0$. So, here in this, $a < 0$ and $b > 0$. So, therefore e^{at} will be equal to 0 as limit t tends to infinity. So, as time goes to infinity e^{at} would become 0 therefore, this vector which is along this axis which I am representing by this will be the solution as I go to as time t tends to infinity.

So, this is going to be a stable line whereas, limit t tends to infinity e^{bt} will tend to infinity that means, what my system is going to diverge along this curve, along this curve. And then what I see is that my system of equations is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and if I want to determine the equilibrium solution, then my equilibrium solution will simply be given as this is equal to 0, which satisfies x_1 equilibrium x_2 equilibrium to be 0 and 0. It is not very difficult to solve this system of equation.

So, my equilibrium solution for this system is this point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and then what I see is that when along the x_1 axis the system converges and along the x_2 axis the system diverges. So, how do I draw the direction of the time? So, the problem in this particular diagram is that there is no time there is only x_1 and x_2.

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So, let me draw the phase portrait elegantly, I have this x_1 axis which is stable. So, therefore, I can see that wherever I start from, I am always going to come towards this point $\begin{bmatrix}0\\0\end{bmatrix}$ because e^{at} for $a < 0$ is going to converge to $\begin{bmatrix}0\\0\end{bmatrix}$. And this is exactly the opposite case for the other axis. So, therefore I would have the direction of the time being denoted like this.

Now, this was the case when C, one case when C_2 was 0 in the other case C_1 was 0. Now, when I introduce both C_1 and C_2, what is going to happen is that I am going to go into one of these quadrants either this or this or this or this depending upon the values of C_1 and C_2 that I choose. So therefore, when this happens when C_1 and C_2 are non-zero, and depending upon whether they are positive or negative, I am going to get one or the other points in these four quadrants.

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And therefore, my solutions would look like this. I would be present in one or the other quadrants. So, these are the phase lines. So, if I start at any one point on this phase line, which is going to be the solution, how am I going to evolve in time, so I need to give the direction of the time on this plot. And that is not very difficult, what I see is that along the x_1 axis, the system is going to be convergent and along the x_2 axis the system is going to be divergent.

So, therefore, the direction of time is this, for this quadrant, the direction of time would be this for this quadrant. Similarly, it would be this in this quadrant and this in this quadrant. The arrows I must repeat give you the direction of time, because the plot has been drawn between x_1 and x_2. So, when you go along the direction of the arrows, you are basically evolving the system in time and therefore, what it basically says is, what this phase portrait says is that, if you start with any one point along the x_1 axis, you are going to converge so therefore, the value of the variable x_1 is going to go to 0, and the value of x_2 is going to diverge to infinity.

Now, imagine a case where, instead of this we wrote here, $a_1 < 0$ and $b > 0$, I have the exact opposite case where $a > 0$ and $b < 0$. What is going to happen in that case? So, this is $a > 0$, so, this axis will become the unstable axis in the second case, and here b becomes less than 0.

So, this axis now will become stable axis and I then leave this as an exercise for you to determine the direction of the arrows in fact, all the arrows will have exactly the opposite direction and what kind of phase portrait is it called this phase portrait is called saddle solution. So, why is it called saddle? Because from one direction you go towards minima from the other direction you go to the maximum, maximum means you go with the system diverges to infinity. So, therefore, it has a saddle like behaviour and such solutions are called saddle solutions.

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Now, we take another case, where you have our equation is this,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And now you have $a > 0$ and $b > 0$ and you can write $a > b > 0$. So, my solution is given by the same equation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{at} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{bt} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The equilibrium solutions remain the same.

So, therefore I will have the same point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as the equilibrium solution and the axis would be x_1 and x_2 but now, what is going to happen is that $a > 0$. So, limit t tends to infinity e^{at} would tend to infinity and limit t tends to infinity e^{bt} would also tend to infinity, remember in the previous case one of them tended to 0 the other one tended to infinity, that is why we call the solution as saddle solution.

Now, when this happens, both the solution so, we had these, we had marked this solution axis and now both the axis are unstable, both axis are unstable and how do I draw the phase lines? This is the question. So, this infinity the system goes to infinity which means the system is divergent, but how do I show this divergence, this is the question, for that I need to do one thing this particular system of equations is such that you can write the solutions of x_1 and x_2 in explicit form.

So, if you solve for x_1 and x_2 using equation 1 what you will get is that x_1 is $C_1 e^{at}$ and x_2 is $C_2 e^{bt}$, I do not know the slopes of the curves which would be present in the four quadrants this is the problem, so then what I would do? I would play a trick from here I would write dx_1 is equal to x_1 is $C_1 e^{at}$ and x_2 is $C_2 e^{bt}$ from where I can write

$$\frac{dx_2}{dx_1} = \frac{C_2 b}{C_1 a} e^{(b-a)t}$$

Now, I have some idea about the dependence of the slope on time I have some idea about the dependence of the slope on time. So, as t tends to infinity what happens to the slope you see here you have *(b – a)* and in our case b < *a* which means (*b – a)* is negative which means that the slope as time t tends to infinity is going to diminish to 0 the slope should become 0 this is what I learned from this expression.

So, how do I draw a set of curves such that as time t tends to infinity the slope becomes 0 and at the same time there is no t anywhere here on the phase portrait. So, let us see if the curve which I draw satisfies this condition. So, if these are two curves on the phase plane, then if I draw a line like this that the direction of arrows like this, then what is going to happen? The direction of the arrow gives me the direction of time and the slope is tending to 0 here as t tends to infinity in this case, in this case, in this case and in this case, in every case.

So, therefore, as time t tends to infinity the system is the system diverging. Is the system going to infinity? Yes, this system is going to infinity, but is it also satisfying the condition that the slope must converge to 0? Yes, it also satisfies the condition that the slope must converge to 0. So, now I can draw certain other lines as well these all are supposed to meet at 0 by the way.

So, these would be the solution curves where they all emerge from $\begin{bmatrix}0\\0\end{bmatrix}$ and then they go to infinity satisfying the condition that the slope at time t tends to infinity tends to 0. And if I follow these curves, do I get the correct nature of these unstable curves? Well, you see, all the curves are pointing out so therefore on this x_1 axis also I will point it out on this y, x_2 axis also I will point it out again along positive x_1 axis I will point it out and along negative x_2 axis I will point it out.

And since I have written that x_1 and x_2 axis are unstable axis, in fact, I should have drawn the arrow out to outward which is actually satisfied in this condition also. So, now since you start from $\begin{bmatrix}0\\0\end{bmatrix}$, and you emerge out these solutions are called source solutions, these solutions are called source solutions. So, we had case 1b where you get source solutions because you had the eigen values, both of which were positive.

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Now, you can do one trick, you can have a situation so let us quickly draw this, we had $a > 0$, we had $b > 0$, and you drew these as the solutions, and all of them were pointing out, this is x_1 is x_2. And the condition was that $a > b > 0$. Now, what is going to happen when $a > b > 0$, but $b > a$, in which case, we will do the same analysis for the gradient for the derivative.

And then in this case, I am pretty sure you will be able to figure out that you will get the phase portrait like this you have pointing out, arrows pointing out, arrows pointing out, arrows pointing out and now, as time t tends to infinity, the slope should become should tend to infinity please do these calculations by hand and if you find the derivative dx_2/dx_1 $e^{(b-a)}$ term would appear and since (b – a) is positive, the slope should tend to infinity.

So, how would I draw the curve which would show that the slope is going to infinity? Please convince yourself that these are how they would look like. And again, this would be the direction of the time. So, this is also a source solution.

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Finally, we have the third case, where both of the eigenvalues can be identified as negative, so dx, so we have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and now we have $a < 0$ and $b < 0$. What is going to happen? Well, you will have the solution as,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1\, e^{a\,t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2\, e^{bt} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, therefore, like the previous case what you would find is that the axis x_1 and x_2 in fact are the solutions.

This axis are the solutions, this point is the equilibrium solution. And now the question is what happens as a limit t tends to infinity e^{at}. When $a < 0$, this is equal to 0 and similarly, limit t tends to infinity e^{bt} when $b < 0$, this is also 0, which means that both the axis now x_1 and x_2 are stable axis. And if x_1 and x_2 both are stable axis I can draw the arrows like this.

They are all converging to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, I will again ask the same question that what would happen to the quadrants which are to the regions which are away from the axis, you can repeat the exact same analysis the solution being given as $x_1 = C_1\, e^{at}$ and $x_2 = C_2\, e^{bt}$, which means

$$\frac{dx_2}{dx_1} = \frac{C_2 b}{C_1 a} e^{(b-a)t}$$

And then for the case when b > a, for the case where b > a, the slope would tend to infinity $\frac{dx_2}{dx_1}$ would tend to infinity. So, how would I draw such curves where the slopes tend to infinity. Now, in this case the direction of the arrow is inward therefore the direction of the time is inward. So, here in this region the slope should become infinity, this is very different from the previous case.

At the centre or at the origin the slope should become infinity, and you see here if I draw the curve like this here the slope is infinity. This is exactly opposite of what we saw in the previous case. So, the direction of arrow would be this and can draw several of such curves. And then what would be the case when a is greater than b? Not very difficult to figure out they would be directed like this, like this. This being the direction of the arrow and so on.

So, what is the nature of the solution. Such solutions are called sink solutions. So, what we saw today is that you can cast a second order dynamical autonomous system in $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$ $\frac{dx}{dt} = Ax$ where $\underline{x}$ is the vector dynamical vector x, $\underline{\underline{A}}$ is the matrix, you can determine the eigenvalues and eigenvectors and for the case where the eigen, where the vector is, the matrix is of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, only for such a case what you can do is you can determine the solutions $C_1 e^{at} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{bt} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

What is going to happen is, now the nature of your solutions will depend upon the values rather the signs of a and b, when one of the eigen values is positive and others eigen value are negative you get saddle solution where one of the axis is going to be stable axis the other axis is going to be an unstable axis. When both of the eigenvalues are greater then 0, they are positive, the system is divergent, both the axis would be unstable axis and you will get source solution.

Similarly, you will get the exact opposite result when both the eigen values are negative both the axis will become stable axis and the solutions would be called the sink solutions. We will take the case of completely different form of the matrix a in the next lecture and see how this can affect the nature of the solutions, thank you.