Advanced Process Dynamics
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Lecture 60
Review of the course

Hello and welcome back. Today we will have the last lecture of this online certification course. We have been meeting for the past 12 weeks, I hope, we learned a lot about some advanced techniques for studying process dynamics.

So, what we will do today is we will recapitulate everything which we studied in this particular course. So, if you remember, we started off this course by formally defining what is the meaning of dynamical systems. Very quickly, a dynamical system is a system which involves quantities which change with time and we find dynamical systems all over around us not only specifically in process industries, but also in day to day life. So, we said that there are basically two ways of analyzing dynamical systems. The first approach is to analyze the system in state space domain. The second approach is to analyze the system in transform domain, what was the major difference in the approach.

The major difference was the motivation, why you want to study the dynamics. Before you implement a control system, you go ahead and run your process plan. You would ideally like to know everything which can happen to your system if I have a reactor for example. What are the different conditions which the system can sample and under different conditions what all can happen to my system? If this is the question, then I emphasize that you must go for the state space domain approach where state space means that you would have a way to understand how different states of the system are sampled. And by state of the system, I simply meant the value of the dynamical variable.

The dynamical variable in general will lie in a linear vector space, and therefore, it would make a space and that is why the approach is called the state space domain analysis. So, we started our course by the state space domain analysis. In fact, if I am not wrong, we studied the state space domain analysis for nearly 8 weeks because this is something which is completely new and in general, it is not τght in the undergrad curriculum. So, we formally define the order of a system in the state space domain analysis and our new definition of the order of a system was rather than defining the order of the governing ODE.

We said that, it is always possible to convert an nth order ODE to n first order ODEs and therefore, we will always define the order of a system as the number of first order ODEs, which go on your system. What was the advantage? The advantage was that by adopting this method, it was possible for us to analyze everything in terms of matrix algebra. So, we very extensively used linear algebra to study the dynamical features or dynamical behaviour of different complex systems, which were even higher order.

So, we started with a simple first order system and learned that a first order system which is autonomous which means a system which is of the form which is described by $\frac{dx}{dt} = ax$ will always have monotonous behavior. You cannot have non monotonous behavior in such first order systems. We also came across a very interesting concept of bifurcation, where we said that, there can be a bifurcation parameter in your system whose magnitude whose sign rather will govern the fate of your system and what did we mean by fate of a system?

Well, the value of the dynamical variable as t tends to infinity, whether you are going to settle to some value or you are going to blow up to infinity and that was dependent upon the bifurcation parameter in the equations of the type $\frac{dx}{dt} = ax$. So, when a was greater than 0, we saw the system blows up to infinity. You have unstable system and when a is less than 0, the system settles to 0. You have a stable system. But then we also define equilibrium solutions. Equilibrium solutions, where the solutions which was obtained by setting $\frac{dx}{dt} = 0$.

So, in any case, if your equation is given as $\frac{dx}{dt} = f(x)$, then all the solutions are all the zeros of *f(x)* are your equilibrium solutions. And what happens at equilibrium solution? if you attain exact equilibrium solution, then the system does not change with time, which means that at every instant farther to the attainment of equilibrium solution, the equilibrium the variable of the dynamical variable would have the same value, which is the equilibrium solution, but equilibrium solutions could be of two types stable or unstable.

We saw that when you have a stable equilibrium solution, that you symptomatically reach the value of the equilibrium solution and then the system remains there. Whereas in case of unstable equilibrium solution, as you move ahead in time, your system goes farther away from the equilibrium solution. So, these were the features which we understood for first order system, but these were actually good enough for letting us explore the higher dimensions.

So, we went ahead and defined a general nth order system and a general nth order autonomous system was defined by a matrix equation d / dt of the vector x, the dynamical vector x is equal to the matrix A double under bar times the dynamical vector $\underline{x}$. What was the general solution? The general $\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$ given , where N is the number of components of your dynamical vector basically order of your system.

So $\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$ where $\underline{V}_i$ under bar where your corresponding eigen vectors. λ_i's were your Eigen values and the stability of your system not dependent upon the eigenvalues of the system. So, for a planar system, two dimensional planar means two dimensional where your dynamical vector has two components only, we saw that there are six different cases which can arise. So, we wanted to draw the phase portrait. So, phase portrait in this case, give us the variation of one component of the dynamical variable versus the other component.

In the first case, you had the solutions such that one eigenvalue was positive and the other eigenvalue was negative, both of them being real. This case, you saw that the face portrayed confirmed to the saddle solution. One of the axis was stable. The other axis was unstable. So, therefore, you have conditional stability. But overall, the system was unstable because the component arising from the eigenvalue which was positive would always blow the system up to infinity.

We had sink solutions when both the eigenvalues were real and less than zero, so you settle to your equilibrium solutions. That is why the term sink. You diverge away from the equilibrium solutions as a source. That is why getting a source phase portrait or source solutions when both the eigenvalues were positive. Now, just these three saddle, source and sink were the cases where you had the eigenvalues were real. What about the case when the eigenvalues were complex numbers? So, when the eigenvalues were purely imaginary, we saw that you have center solutions. So, x_1 and x_2 keep on oscillating about the same point and they keep on repeating in time.

The magnitudes of the amplitudes of oscillations to not change for purely imaginary Eigen values. So, this was the case when you had only when you had the center solution and purely imaginary Eigen values, when the Eigen values were complex such that the real part was positive greater than zero then you had spiral source solution. You start from the center and you move away diverge to infinity. Conversely, you spiraled down to your equilibrium solution in case when you have the real part of your eigenvalues negative.

Now, this analysis put us in a fair position to analyze higher order systems as well. So, for example, so, the cases of third order system where the saddle solutions could be fragmented into subspaces, where you could have a purely stable subspace or purely unstable subspace and so, on. We use the same concept and extended it. For all of this analysis, which we took the examples of cooling of a body for example, was the first order system. We studied the dynamics of oscillations of vibrations in spring mass system both forced as well as free.

In force systems, we had similarity solutions, so, we develop the concept of similarity solutions. And this was the case for non-autonomous system where your dx/dt could not simply be defined as f(x), but it was defined as f(xt). Then, we also took the case of complex reaction kinetics where you had higher order dynamics and we saw how the evolution of different concentrations A, B, C in your system could be very nicely described as following the principles of dynamics.

So, this is more or less what you studied in case of linear dynamics. In all these cases the system was linear, because the system followed the principle of linearity where you identify the operator, you identified two solutions and some of the two solutions operated by the operator was some of individual solutions, operated by individual operator and if you multiplied the solution by a constant that constant comes out and then you had the operator operating on the solution back. This was the simple principle of linearity. When you do not have a linear system, then you basically have two approaches.

We studied that one of the processes solving the problem directly in the nonlinear domain. In certain cases, it is possible. We took very different cases of population dynamics. We studied the logistic equation, we study the logistic equation with harvesting, we studied the logistic equation with critical threshold, we develop the phase portraits, we saw how without explicitly solving these equations, we can develop the phase portrait and get a qualitative idea of the solutions, the equilibrium solutions, whether the solutions are stable or unstable, whether there is a bifurcation in this system and so on.

But then we said that, it is not always possible to solve the problem in nonlinear domain. So, at times you might need to linearize your model. So, how did you linearize the model? Using Taylor series expansion, one of the ways. We took even a trivial way where you just made the non-linear term zero that was the case when you always had 0, 0, 0 an origin as the equilibrium solution, but then we assembled upon one question that how do you make sure that linearization would always work?

In other words, when you have a nonlinear phase portrait, you had the orbit structure and you had the linearized version and you had the orbit structure of that, how do you know that both of them are same? Well, both of them cannot be the same, but perhaps they would be the same about the equilibrium point. So, if you know the equilibrium solutions, probably you can consider a case where the orbit structures about the equilibrium points are the same. How would you know that? This resorted to Hartman-Grobman theorem, according to which we saw that we will first determine whether the equilibrium solution is a hyperbolic solution or not.

What is hyperbolic solution? The corresponding Eigen values should not be zero, none of the Eigen value should be zero or the real part should be zero in any of the cases. If this condition is satisfied, then the equilibrium solution is called hyperbolic and under such condition you say that okay probably under such condition only the orbit structures close to the equilibrium solutions would be same that gave us a confidence that fine if I can establish the hyperbolicity then I can say that as long as I do not move pretty far away from my equilibrium solution.

The qualitative nature of the solutions at least would be the same. Then we made use of this fact to do reactor stability analysis, we saw how different steady states can be achieved in a dam during a diabetic operation of a CSTR which ones will be stable which one would be unstable and the reason behind this stability or instability. We also saw different types of bifurcations in first order nonlinear equations. For example, we saw pitchfork bifurcation, we saw transcritical bifurcation, these are the different type of bifurcations which we came across.

So, this is what we learned during the state space domain analysis. Then we realize that in process industries, these days control systems are in place and therefore, it is very likely that you would be moving between two steady states and probably you would like to come back to the same statistic again. Therefore, you are very often interested in determining the dynamical behaviour of the system between two steady states. And if that is the case, what you do is you do the analysis in transform domain, we said that the system can be continuous, the system can be discrete time.

In fact, we did the discrete time analysis in state space domain analysis as well invoking the concept of fixed points. So, when we did the analysis of continuous domain systems in transform domain, we adopted the method of Laplace transform. So, we always try to develop an input output model and then we try to define the transfer function. Transfer function was a key quantity, it was the key quantity and transfer function was given as the ratio of the Laplace transform of the output function in deviation variable form divided by the Laplace transform of the input function in the division variable form.

What was the advantage that we got by doing this analysis? By doing this analysis, we were in a position to get the response of the system subject to various forcing functions. We identified various ideal forcing functions. For example, the idea is step input function and ideal ramp functions, sinusoidal function, rectangular pulse function, impulse function, so on. Now, if I know my input function then I can correspondingly determine the Laplace transform and then what I would do is I would multiply my input function Laplace transform by the transfer function.

What would I get? I would get the Laplace transform of the output variable in deviation variable form. I assume that the time at which the disturbance or the forcing function is given to the system is time t is equal to zero. This makes my life easy and therefore, what I do is now, after I get $\bar{y}$(s), all I need to do is I need to invert my Laplace transform and that would give me y of t. So, what is the difference between y(t) and x(t)? x(t) which we got in the state space domain analysis, y(t) is the response of the system. So, it always starts with zero and describes the response of the system subject to the input function we have provided. We took the case of various responses step response, sinusoidal response, ramp response and so on and try to understand the physical meaning of the time constant and the physical significance of the strategy of the system.

We saw that as we see in case of first order system in state space domain analysis in this case also for the first order system, the response is always monotonous. When we took the case of two first order systems in series, we saw that we got a second order transfer function in this case also, you always saw that the system has had a monotonous response and the response of a higher order system was always slower compared to the responsible lower order system. So, as we kept on adding poles to our transfer function, the response became more and more and more slower.

Then we took the case where we added a zero to the system and what we found was that, if you add a zero to the system the response becomes enthusiastic the response becomes faster as you add a pole to the system the response becomes slower. So, therefore, we defined a general p, order system. So, a p, q order system would have P number of poles and Q number of zeros. So, the general transfer function of a p, q order system will have a polynomial in the numerator, which will have the degree Q and a polynomial in the denominator, which would have the degree P.

We studied the dynamical response of several of the systems and then invoked the concept of transfer function matrix. Well, you can always have more than one input and more than one output. So, for a single input, single output system you define a transfer function which means that you had the Laplace transform of the input, you had the Laplace transform of the output and division of output by input in the division variable form give you the transfer function. This was a 1:1. Now, if you have multiple inputs and multiple outputs, we developed a method which would give us the transfer function matrix. The method of developing this was very similar as what we did previously and we got a P X M matrix.

So, each element of the matrix would give you the individual effect of the transfer function which means, it would give the individual effect of a given input function on rest all of the output functions and so on. So, for multiple input multiple output system, you will need to generate the transfer function matrix. We also saw the case of inverse response systems. So, when you have a transfer function of the form $K_1 / (\tau_1 s + 1) - K_2 / (\tau_2 s + 1)$, then you have two modes, you have the main mode and you have the opposition mode.

Now, depending upon τ_1, τ_2, K_1 and K_2 it was possible that subjecting the system to a positive step may result in the initial response of the system, which goes towards negative and that is why the system was called inverse response system. We were in a position to determine the condition when the system will result into inverse response. And correspondingly, we were also in a position to determine the exact time when this inversion back to the correct direction of your response will take place. We saw that the inversion will happen, but ultimately, the response the system would realize that you are going in the wrong direction it would catch up to the step.

So, this was all which we studied in the continuous domain analysis of the systems under transform domain analysis. We realized then that most of the processes these days involve digital computers for control because when you write the control algorithm, they would ultimately be fed to a computer and therefore, you need the involvement of a computer always during any process operation. Now, what is the problem with this? We discuss that the input signal would be a continuous signal. Imagine that you want to control the temperature of a reactor with the help of a cooling fluid.

So, what you would like to do is that you would like to record the temperature and send error signal so that the valve will open and change the flow rate of the cooling fluid. So, input signal to the error is always a continuous because signal because it is an EMF from the thermocouple. The output also is always continuous. It should be continuous because you want to continuously change the valve position to change the flow rate, but all of this has to be done with the help of a computer, A computer will not accept, a computer will not be able to understand a continuous signal.

So, you first need to convert a continuous signal to a discrete signal. How is it done? It is done with the help of a sampler. So, the first step in the analysis of discrete domain analysis of response of the systems is to convert continuous signal to discrete signal with the help of a sampler which is in a very simple case switch, then the signals would go to a computer and computer only can handle and understand and do any operation and discrete form. So, your model which is continuous must be converted to a discrete time model. We learned how to convert your models to discrete time models.

Finally, the control action which will be provided would be based upon the suggestions by the computer and therefore, they also would be discrete time and you would not like that. So, therefore, you want to convert your discrete output to a continuous output. How do you do that? You do that with the help of a hold element. So, hold element will try to fill in the signal between two discrete signals, give you a continuous time signal.

Now, once you have this entire arrangement ready, you would be interested in knowing the discrete time domain transfer function. The concept that we invoke for this was the pulse transfer function. But in order to analyze anything with pulse transfer function, you need to do z transforms rather than Laplace transforms. So, we studied in detail what are z transforms, what are the inputs to it, what is the output from it and the way we understood Laplace transforms, how to deal with them, how to do Laplace transforms, we did try to understand the study is a transforms. Then we understand here now that we will be using only z transforms for discrete time domain analysis.

So therefore, we said that, let us use the same framework that the meaning of a transfer function is that you have the output, you have the input in the denominator and that would be the transform would be the ratio of the transforms would recall the transfer function, but instead of transfer function, we would call it pulse transfer function to indicate that you are dealing in discrete time domain. So, we understood the methods of developing pulse transfer functions, which would involve no hold as well as hold if you decide to incorporate a hold element.

And finally, the way we used various forcing functions multiplied to the inverse Laplace transform, we did the same analysis, in this case of pulse transfer function. It is inverse, it is multiplication with z transform of the input and inverse that transform of z transform to get the output function.

Finally, we said that we knew a method to analyze the stability of the system in state space domain, can we develop a similar method to analyze the stability and transform domain. In fact, we did, we said that for continuous system, you just need to look at the transfer function. And if the transfer function has all negative poles, or in case of imaginary poles, negative real parts, then the system will be stable. We saw the reasoning behind it. And for the case of pulse transfer function, you make a complex plane, see, draw a unit area and within that, if your pole is inside the area, your system is always going to be stable.

So, it is been a long set of discussions. I hope that you would not only understood the various concepts very well but you also enjoyed it. And by after learning all of these new concepts, I hope that you would be in a better position to not only understand but also to take interest in further courses, advanced courses on process control. So, we stop here, and I wish you good luck in your future endeavours. Thank you.