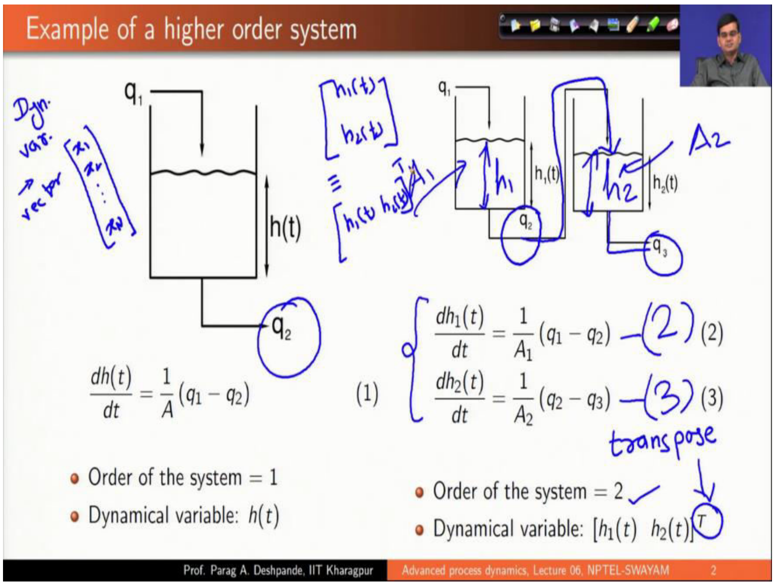
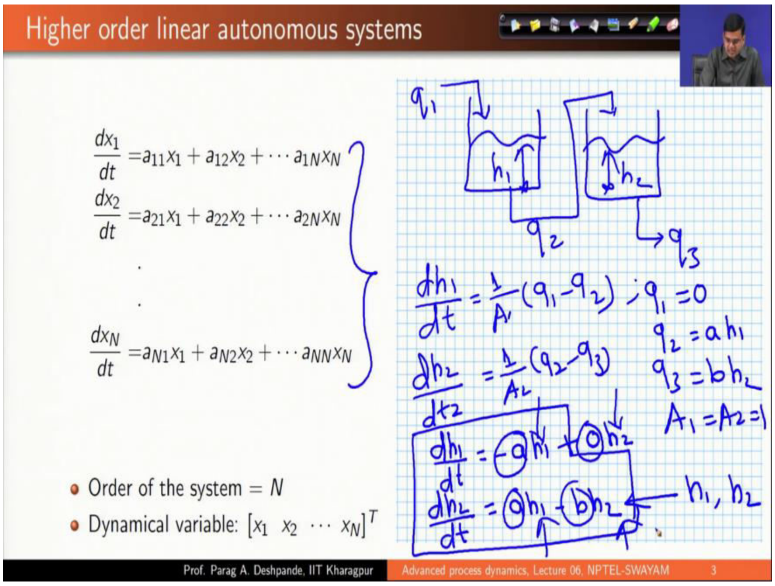
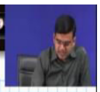
Advanced Process Dynamics
Professor Parag A. Deshpande
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur
Lecture 06
Introduction to higher order systems





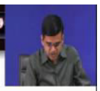
Higher order linear autonomous systems



$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots a_{1N}x_N$$
$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots a_{2N}x_N$$
$$\vdots$$
$$\frac{dx_N}{dt} = a_{N1}x_1 + a_{N2}x_2 + \cdots a_{NN}x_N$$

- Order of the system $= N$
- Dynamical variable: $[x_1 \ x_2 \ \cdots \ x_N]^T$

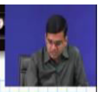
Higher order linear autonomous systems



$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1N} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{N1} & a_{N2} & \cdot & \cdot & \cdot & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} \quad (4)$$

N^{th} order dynamical equation: $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$ $\quad$ 1^{st} order dynamical equation: $\frac{dx}{dt} = ax$

Higher order linear autonomous systems



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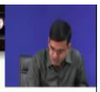
Higher order linear autonomous systems



$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$

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vector : $\underline{x}$

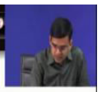
matrix : $\underline{\underline{A}}$

$x \longrightarrow \underline{x}$

$a \longrightarrow \underline{\underline{A}}$

N^{th} order dynamical equation: $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$ ✓ 1^{st} order dynamical equation: $\frac{dx}{dt} = ax$ ✓

Solution of N^{th} order linear autonomous equatic



Theorem

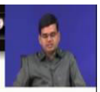
The solutions to a linear autonomous equation of the form $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$ ($N \times N$) are given as

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

(c_i: constant multipliers; λ_i: eigenvalues; $\underline{v}_i$: corresponding eigenvectors)

where,
λ_i's are the eigenvalues of $\underline{\underline{A}}$
$\underline{v}_i$'s are the corresponding eigenvectors
c_i's are present in the field over which the vector space of solutions is defined

Solution of N^{th} order linear autonomous equatic



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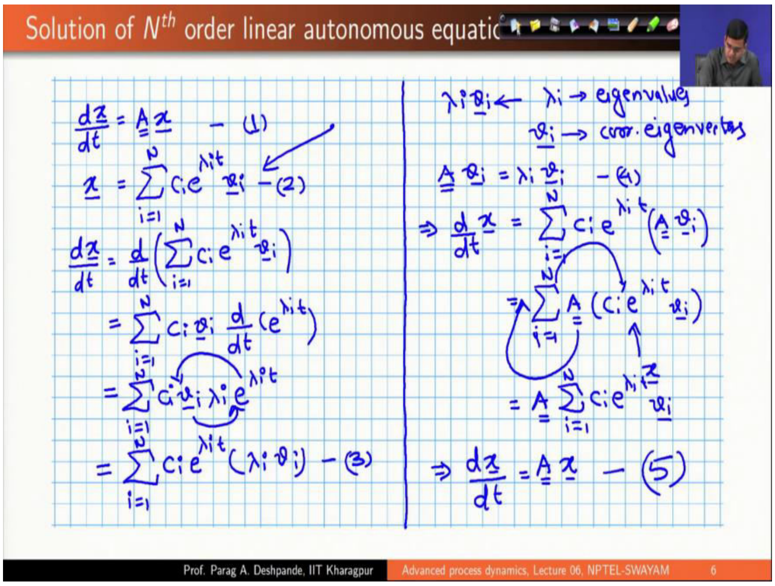
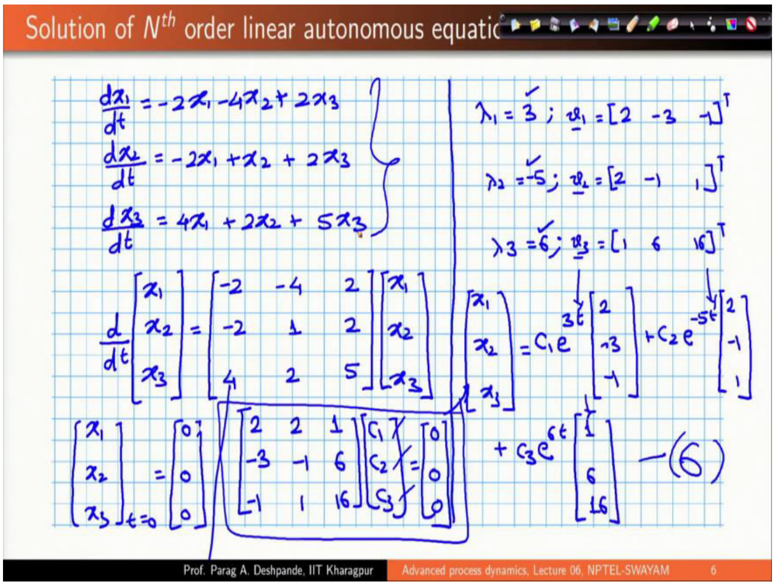
Prof. Parag A. Deshpande, IIT Kharagpur Advanced process dynamics, Lecture 06, NPTEL-SWAYAM 5





Hello, and welcome to the second week of instruction of this NPTEL online course on Advanced Process Dynamics. In the previous week, we focused on linear systems. And in linear systems, we focused on linear autonomous systems, we took the examples of first order systems and typically saw how the behaviour of first order systems would be.

So, if you model a first order linear autonomous system as $\frac{dx}{dt} = f(x)$; where $f(x) = ax$ to make the system autonomous, then your solution will be of the form x_0, where x_0 is the initial condition multiplied by e^{at}.

So, $\frac{dx}{dt} = ax$ or a similar variant of it will have the dynamics which will follow $x_0\ e^{at}$ or a similar variant. We saw that these dynamics will always be monotonous which means that depending upon the value of A, the dynamical variable $x(t)$ would either continuously increase with time thereby diverging to infinity or it would always decrease with time thereby converging to a constant value as time t tends to infinity.

In this week, we will take second and higher order systems, we will realize that higher order dynamics are inherently difficult as well as inherently different in certain features when compared to first order dynamics. So, let us see and how we can model second and higher order dynamics and where do we come across such situations.

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On the left-hand side, you see the example of water level in a tank which we considered in the previous week, where there was one dynamical equation given by

$$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$$

where q_1 is the input flow rate, q_2 is the output flow rate, A is the area of cross section of the tank. The order of the system is 1 because we have one dynamical equation and the equation is first order ODE and the dynamical variable in this case was *h(t)*, now you imagine that the output from the first tank, this one is fed to the second tank.

So, in the figure on the right-hand side, you have the output from the first tank and this output is fed as an input to the second tank and you have the ultimate output from the entire system as q3. So, now, we can write similar balance equations as

$$\frac{dh_1}{dt} = \frac{1}{A_1}(q_1 - q_2) \qquad \dots (2)$$

$$\frac{dh_2}{dt} = \frac{1}{A_2}(q_2 - q_3) \qquad \dots (3)$$

Now, you will see that we have equation 2 here, which is a first order ODE and equation 3 which is also a first order ODE and altogether equations 2 plus equation 3 describe the dynamics of the system. Therefore, the order of the system is 2 in this case. So, we have actually obtained a second order system by considering two simple first order equations.

But in this case, the dynamical variable is represented as $[h_1(t)\ h_2(t)]^T$, the superscript here represents the transpose. So, let me write here transpose. So, you have a vector. So, we will define a vector as a column matrix. So dynamical variable would be a vector, and what would be a vector in our case, it would be a column matrix.

So, matrix have the form x_1 x_2 and for an n^th^ order system, up to x_n. So, this in fact is a matrix, but this is a column matrix and therefore, we would refer this column matrix as a vector and the dynamical variable in such a case would be simply a vector. So, for the case given here you have a dynamical variable which is a vector given as a $\begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$ or alternatively $[h_1(t)\ h_2(t)]^T$.

So, there are two ways of writing this you can try this as $\begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$ which is equivalent to writing $[h_1(t)\ h_2(t)]^T$. So, this will be our notation for representing dynamical variables in higher dimensions.

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So, now we can write a general n^{th} order equation or n^{th} order system dynamical system by simultaneous equations which are given here. Now, what is the genesis of these equations, let us consider the previous case where you have the input q_1, input q_1, output q_2 goes as the input to the second tank and you get the output q_3. So, this is h_1 and this is h_2.

Now, our dynamical equations for $\frac{dh_1}{dt} = \frac{1}{A_1}(q_1 - q_2)$; So, let us consider a case where q_1 is 0 and q_2 is simply a linear function of h_1. So, you have a wall which would vary the flow rate which causes ah_1. And we have $\frac{dh_2}{dt} = \frac{1}{A_2}(q_2 - q_3)$ and q_3 would go as bh_2 and for the ease of representation let us consider unit cross sections of both the tanks so, I can write here $\frac{dh_1}{dt} = -ah_1$ and $\frac{dh_2}{dt} = ah_1 - bh_2$.

So, we have these two equations and what we see is that we have the equations which are coupled. So, this equation for example, has h_1 and h_2 both so, equations are coupled in fact, you can always write this equation as plus 0 times h_2 and you will see that now you have converted the set of equations into individual equations where you have coefficients, you have coefficients and you have the components of your dynamical variables. So, therefore you can write a general n^{th} order system as a set of n first order ODE's which would be coupled.

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And therefore, the order of the system would be n, the dynamical variable would be this vector $[x_1\ x_2\ \dots\ x_n]^T$ and you will give the individual equations as $\frac{dx_1}{dt}$ which will be some linear combinations of the individual weightages of the individual components similarly, for $\frac{dx_2}{dt}$ and so on. Now, if this be the case, I can convert this set of equations to a matrix equation.

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So, this is how the matrix equation looks like. So, how would you get this matrix equation? Well, it is not very difficult. So, our original set of equations was this. So, how do I write this?

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I can write this as $\frac{d}{dt}$ and I will consider all these components of the dynamical vector. So, I can write this as $\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ and this would be equal to a matrix multiplied by the dynamical variable, original dynamical variables. So, x_1, x_2 x_n. So, this would be x_1, x_2 up to x_n and then the coefficients here a_{11} a_{12} and so on would go as the elements of the matrix, so, a_{11} a_{12} up to a_{1N} up to a_{21} a_{22} up to a_{2N} and so on, so this will be a_{N1} a_{N2} up to a_{NN}.

So, this is the matrix form of your dynamical equations. So, dynamical system which is of general n^{th} order can be expressed in terms of a matrix equation where the dynamical vector will have n components and the corresponding matrix will would be an N x N matrix.

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So, let us see how does the equation looks like, the equation looks like this $\frac{d}{dt}$ of the dynamical vector is equal to the matrix multiplied by the dynamical vector and we will use this notation consistently throughout the course, where the vector would always be represented by an under bar, a single under bar and a matrix would be represented by two under bar.

So, therefore, this would be represented as $\underline{x}$ and this matrix would be represented by $\underline{\underline{A}}$. So, therefore, I can write this equation a $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$ and how do I compare and contrast this

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$$

equation with a first order dynamical equation and nth order dynamical equation is given by the

and the first order dynamical equation is simply given as $\frac{dx}{dt} = ax$.

So, the functional form of these two equations is in fact identical, all you need to do is for a first order system when you have x as the dynamical variable you need to change it to $\underline{x}$ to indicate that this is a vector this is an n^{th} order system and the variable A which you have in a first order system has to be converted to a matrix $\underline{\underline{A}}$ and therefore, now you can do us an analysis to find similarities between these two systems and also contrast the differences between these two systems.

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So, let us look into the details of how to handle such a system. So, we solve the equation $\frac{dx}{dt} = ax$, the first order dynamical equations simply as $x_0\, e^{at}$. How do you solve an n^{th} order system or how do you solve a matrix equation? That is solved using this theorem. The theorem says that the solutions to a linear autonomous equation of the form $\frac{dx}{dt} = ax$; in fact this is the equation which we are currently dealing with. We want to find the solution. It is given as

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

where N is the number of components in your system, C_i are the constant multipliers, constant multiplication factors or multipliers. In fact, one would realize that these are the members of the field over which the linear vector space of the solution is defined, multiplied by $e^{\lambda it}$, t very clearly is the time, but λ_i's are the eigen values, eigen values and $\underline{V}_i$'s are the corresponding, it is very important, corresponding eigen vectors, this is important that you make absolutely sure that when you use Eigen values for your matrix which matrix by the way this is the matrix A.

So, you have an N x N matrix in your system. So, you would determine the eigen values of that N x N matrix A, you will determine the corresponding eigen vectors and then your solution

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

would be given as ; Can we see whether this is in fact the case always? Let us see.

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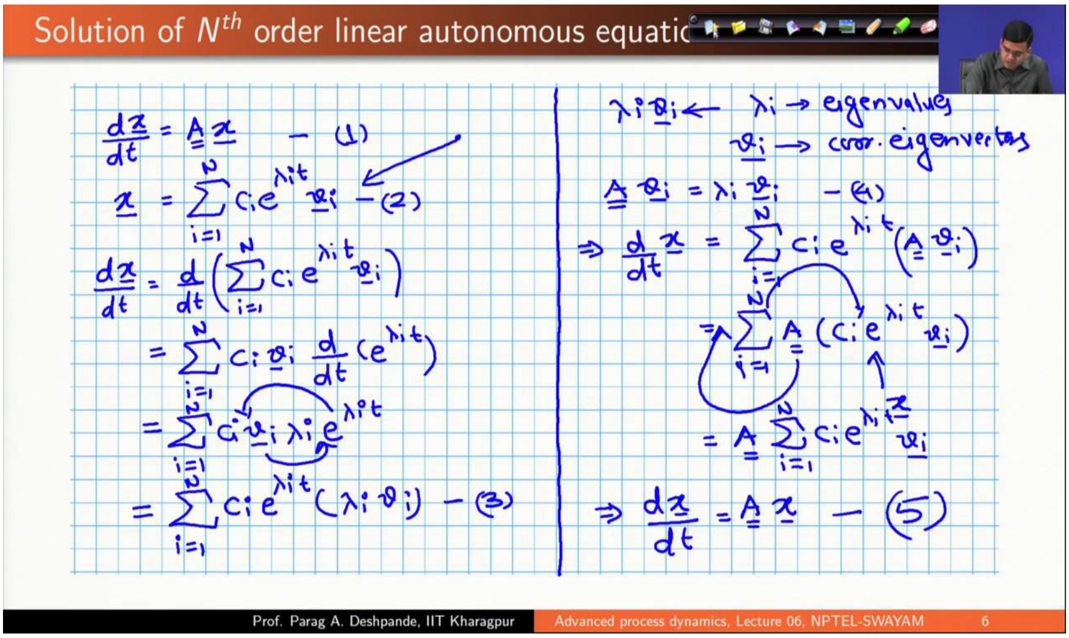
So, we have the equation which is given as $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$;This is the equation that we have. Now, the solution that we have is given as

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

Now, if x is the correct solution to this equation 1 then if I substitute *x* on the left hand side and I substitute *x* on the right hand side then I should get identical expressions. so, we can test this so, if equation 2 is the correct solution then



So, now how can this entire concept be probably used for solving a problem?

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Well, we can actually solve a dynamical equation for example, I have a system of equation in front of me. So, the equation is

$$\frac{dx_1}{dt} = -2x_1 - 4x_2 + 2x_3$$

$$\frac{dx_2}{dt} = -2x_1 + x_2 + 2x_3$$

$$\frac{dx_3}{dt} = 4x_1 + 2x_2 + 5x_3$$

So, this is the dynamical system that I have, I have a third order system. I have three first order equations, these equations are linear, these equations are in fact autonomous, it is not very difficult to realize that these equations are autonomous and these equations are coupled you cannot solve for x_1, x_2 and x_3 individually.

So, I will need to resort to the method which I developed just now. So, my equation can be transformed to a matrix equation which would be of the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So, now what do I need to do for solving this equation, I need to determine the eigen values and eigen vectors of A.

So, in fact, I have those eigen values and eigen vectors already determined with me. So, the eigen values $\lambda_1 = 3$, $\lambda_2 = -5$ and $\lambda_3 = 6$ and the corresponding eigen vectors are $v_1 = [2\ -3\ -1]^T$ $v_2 = [2\ -1\ 1]^T$ and $v_3 = [1\ 6\ 16]^T$. I assume that you still know or remember that the method to calculate the eigenvalues and eigenvectors of a square matrix, please go back and try to recapitulate the method in case you have forgotten.

So, this is the eigen set of eigen values and eigen vectors and therefore, I can write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{3t}\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{6t}\begin{bmatrix} 1 \\ 6 \\ 16 \end{bmatrix}$$

This is my solution.

Now, what I do not know yet is the multiplicative constants C_1 C_2 and C_3 but those are not very difficult to determine for that I will need the initial conditions. So, let's say that the initial conditions are given to me as x_1 x_2 x_3 at t is equal to 0 says 0 0 0 just for an example. So, what will happen I will substitute this here and I will substitute t is equal to 0 in all of these cases. So, it is not very difficult to see that what you will get is an equation of the form

$$\begin{bmatrix} -2 & 2 & 1 \\ -3 & -1 & 6 \\ 1 & 1 & 16 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots(6)$$

Now, this is a simple matrix equation which corresponds to a set of linear equations which are coupled and you need to solve it for C_1 C_2 C_3. You can do it by changing it into a upper triangular matrix using Gaussian elimination and then you will, you can determine the values of C_1 C_2 and C_3. So, I leave this as an exercise for you to determine the values of C_1 C_2 and C_3 from this matrix equation and then substitute that in this equation in equation number 6 to get the solution for $\frac{dx_1}{dt} = -2x_1 - 4x_2 + 2x_3$ and so on which is given here.

So, now we have a method by following which we could easily determine the solutions of an n_{th} order dynamical system, which can be cast as $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$ which looks like an eigenvalue problem, which in fact is an eigenvalue problem, this is a linear autonomous n^{th} order system and the solution for this system is given us

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

So, while you can use this method for systems of any order something which is of particular relevance as well as particular interest for process industries is second order systems. Second order systems are not only found amply in process industries as well as in general physics and also in nature, but they also pose a very good example for the study of further higher order, further complex systems. So, what we will do is that, we will take the example of second order systems and we will try to understand the dynamics of second order systems in the rest of the lectures in this week. Thank you.