Advanced Process Dynamics
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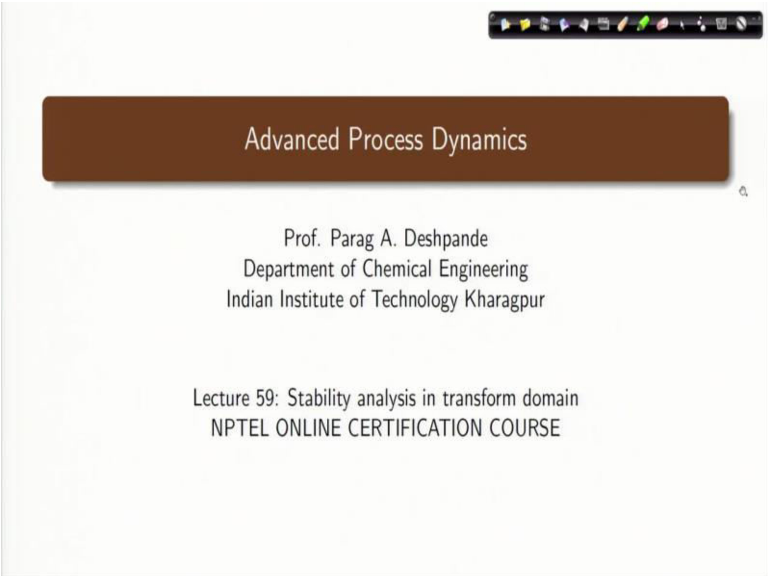
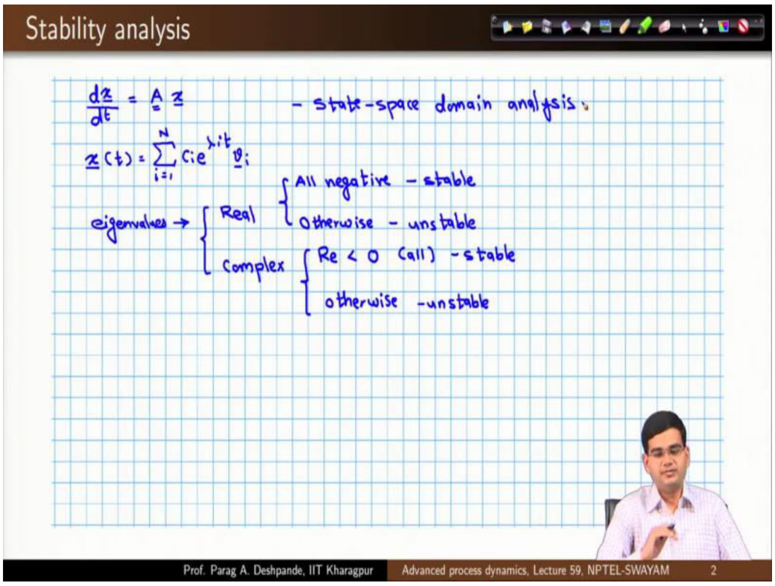
Lecture 59
Stability Analysis in Transform Domain





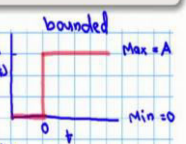
Stability analysis: Continuous systems

Definition

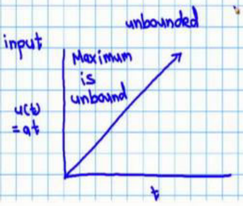
If in response to a *bounded input*, the dynamic output of the system remains bounded then the system is said to be stable.

* bounded input $\begin{cases} \text{Max} \\ \text{Min} \end{cases}$ $\rightarrow$ step input A

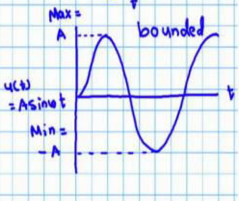
* Response $\begin{cases} \text{Max} \\ \text{Min} \end{cases}$



Ramp input



Sinusoidal input



Stability analysis: Continuous systems

Definition

If in response to a *bounded input*, the dynamic output of the system remains bounded then the system is said to be stable.

Condition

A continuous system is stable if every pole associated with its transfer function is negative or has negative real part.

Stability analysis

$$g_{p,q}(s) = K\left(\frac{a_0 + a_1 s + a_2 s^2 + \dots + a_q s^q}{1 + b_1 s + b_2 s^2 + \dots + b_p s^p}\right)$$

$$\bar{y}(s) = AK\,\frac{1}{s}\left(\frac{a_0 + a_1 s + a_2 s^2 + \dots + a_q s^q}{1 + b_1 s + b_2 s^2 + \dots + b_p s^p}\right)$$

$$= AK\,\frac{1}{s}\left(\frac{a_0 + a_1 s + a_2 s^2 + \dots + a_q s^q}{(s-r_1)(s-r_2)\dots(s-r_p)}\right)$$

$$= AK\left[\frac{c_0}{s} + \frac{c_1}{s-r_1} + \frac{c_2}{s-r_2} + \dots + \frac{c_p}{s-r_p}\right]$$

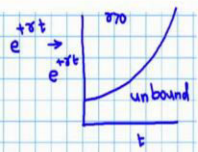
$$\Rightarrow y(t) = AK\left\{c_0 + c_1 e^{r_1 t} + c_2 e^{r_2 t} + \dots + c_p e^{r_p t}\right\}$$

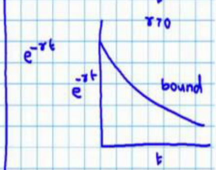
when $r_i \in \mathbb{R}$, $r_i < 0$ for $y(t)$ to be bound

when $r_i \in \mathbb{Z}$, $e^{r_i t} = e^{|r| e^{i\theta}}$ $\quad \theta = \tan^{-1}(\mathrm{Im}/\mathrm{Re})$

$$e^{r_i t} = e^{(z_1 + i z_2)t}$$

$$= \underbrace{e^{z_1 t}}_{\text{decay or growth}}\ \underbrace{e^{-i z_2 t}}_{\text{oscillations}}$$





Decay for z_1

Growth for z_1

Stability analysis: Continuous systems

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Stability analysis

$$\tau\frac{dy}{dt} + y = ku$$

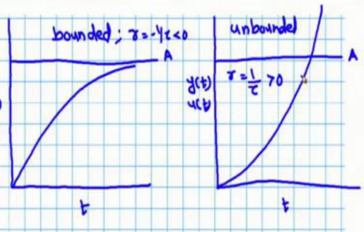
$$\Rightarrow g(s) = \frac{k}{\tau s + 1}; \quad y(t) = Ak\left(1 - e^{-t/\tau}\right)$$

$$\gamma = -\frac{1}{\tau} < 0$$

$$\tau\frac{dy}{dt} - y = ku$$

$$\Rightarrow g(s) = \frac{k}{\tau s - 1}; \quad y(t) = Ak\left(-1 + e^{t/\tau}\right)$$

$$\gamma = \frac{1}{\tau} > 0$$



Stability analysis

$$g(z) = \left(\frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_q z^{-q}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_p z^{-p}}\right)K$$

$$y(z) = AK\left(\frac{1}{1 - z^{-1}}\right)\left(\frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_q z^{-q}}{(1 - \gamma_1 z^{-1})(1 - \gamma_2 z^{-1}) \cdots (1 - \gamma_p z^{-1})}\right)$$

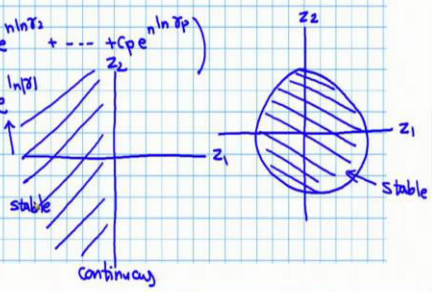
$$= AK\left[\frac{c_0}{1 - z^{-1}} + \frac{c_1}{1 - \gamma_1 z^{-1}} + \frac{c_2}{1 - \gamma_2 z^{-1}} + \cdots + \frac{c_p}{1 - \gamma_p z^{-1}}\right]$$

$$\Rightarrow y(nT) = AK\left(c_0 + c_1 e^{n\ln\gamma_1} + c_2 e^{n\ln\gamma_2} + \cdots + c_p e^{n\ln\gamma_p}\right)$$

$$e^{n\ln\gamma_i} = e^{n\ln\left[|\gamma| e^{i\theta}\right]} = e^{ni\theta} e^{\ln|\gamma|}$$

$$|\gamma| = \sqrt{z_1^2 + z_2^2}$$

$$\ln|\gamma| < 0 \rightarrow |\gamma| < 1$$



Welcome back. So, till Now we were interested in understanding various quantities associated with the transfer function both in transfer in the continuous domain as well as in discrete time domain and its effect on the dynamical behavior. When we say the dynamical behavior, we were basically focusing on the rate of response or the speed of response, one important characteristics of dynamical feature is the stability of the system. Till now, we have not paid pretty much attention on deciding whether the system would be stable or not, when subjected to a given input to the system. So, we will focus on this particular topic in this lecture.

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So, what we have in front of us is a system in state space domain. So, imagine that I have autonomous system which is described in state space domain given us the $\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x}$. So, this is an nth order system, nth ordered autonomous system and we spent a lot of time to understand how to assess the stability of the system apart from the dynamical features of the system and how would you decide on the stability of the system?

Well, you know that the general solution of the system is given us,

$$\underline{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

So, what you basically do is you list out all the Eigen values, Eigen values, and what are the 2 major categories in which the Eigen values may fall? They may either be a real or they may be complex, and depending upon whether they are all real or complex you can immediately see the Eigen values and comment upon the state of stability.

So when the real eigenvalues such that you have all negative every single Eigen value is negative then you definitely see that the system is stable, which means every single state which the system can sample would be bounded, no state will blow up to infinity this is the simple definition of stability and otherwise the system would be unstable. Even if there is one Eigen value which is positive then e to the power positive quantity would blow up to infinity as t tends to infinity.

And when you have complex eigenvalues then you want it to look at the real parts and if the real parts all the real parts less than 0, let me emphasize all real parts are less than 0 negative the system was stable and otherwise the system was unstable. So, this was what you could infer from the state space domain analysis. Look at the Eigen values every single negative Eigen value or in case of complex Eigen values every single real part being negative is assurance of stability and if even one of these is not satisfied the system is going to be unstable, pretty straightforward. Now you have an input output type system where you would like to know the response of a system subject to a given input, and then now you have to define stability. So, you need to define stability and we know that we need to do this analysis in the transform domain.

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So let us see how do we do that? So, the first thing which you need to do is, you need to define what exactly is the meaning of stability. So, if in response to a bounded input, so the first thing which you need to worry about is bounded input, what is the meaning of bounded input? There is a minimum and maximum which has been defined for the input, so there is a minimum and maximum which is defined for the input. So, if in response to a bounded input, the dynamic output of the system remains bounded which means, the response also is bounded between maximum and minimum then the system is set to be stable otherwise unstable.

So if we understand the meaning of bounded input obviously, the meaning of a bounded response would follow, so let us try to understand the meaning of bounded input. Let us imagine a step input of magnitude A. So, what happens here in this particular case? You give your system, so this is u of t and this is t, this function is defined such that you have the value 0 and a value a at t is equal to 0 which is sustained there, so this is 0 this is A. So, what is the input what are the bounds on the input? Minimum is 0 and the maximum is A, so there are bounds. So, therefore, a step input follows the condition of bounded input, a step input is a bounded input.

What about a sinusoidal input for example? Input, so I have here t and I have here

$$u(t) = A \sin \omega t.$$

What is going to happen in this case your function is going to look like this. So, what is the maximum that you have? It is A, this is the maximum and the minimum that you can have is minus A. So, again, your input is bound between 2 limits the upper one being plus A, the lower one being minus A. So, therefore, sinusoidal input also is an example of a bounded input.

What about a ramp input. So, I write here bounded, bounded and I have a ramp input now, what is the functional form of a ramp input? U(t) is equal to a t and versus t. You see here the input goes to infinity as t tends to infinity. So, therefore, maximum is unbound there is no upper cap on the maximum and therefore, you have an unbounded input. So, therefore, you may not say anything or you may not be in a position to comment upon the stability of your system, if you subject the system to a ramp input, but you may subject the system to a step input and you know that the maximum that my input or my forcing function can have is A, the minimum that it can have is 0. So, therefore, if my response also is bounded then the system is called to be stable.

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So now let us see how we can understand the dynamical behavior with respect to the stability of the system. The condition for stability is that a continuous system is stable if every poll associated with its transfer function is negative or has a negative real part. So, this means that I can simply look at the transfer function and say that I know that this system is going to be stable, I know this system is going to be unstable, the way I used to look at the Eigen values in the state space domain analysis and the moment I used to see that the Eigen values are positive, I use to say that a system is going to be unstable, when I saw all Eigen values to be negative as to say that the system is going to be stable.

When the Eigen values used to be complex numbers with all negative real parts and I used to know that the system will have oscillations, but oscillations will eventually die out and when they had positive part I used to know that there would be oscillations and the oscillations would increase in amplitude with time, the oscillations would sustain if you have you had a purely imaginary eigenvalue. So, similarly, now what I can do is I can look at the transfer function, the Laplace domain transfer function in case of continuous systems and I would look into the poles of the system. And if every pole associated with the transfer function is either negative or the real part is negative, if it is a complex pole, then the system will be stable. Let us see if we can establish this.

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So, imagine that I have a p, q order system. So, let us take the most general case. So, I have,

$$g_{p,q}(s) = k\left(\frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_q s^q}{1 + b_1 s + b_2 s^2 + \ldots + b_p s^p}\right)$$

this is the numerator having a polynomial of degree q divided by an polynomial of degree p. So, this I am considering the most general case. Now if this be the case, I would like to know the dynamical response and the input itself has to be bounded, so therefore, let us take the case of a step input. One thing which I have forgotten is that I need to multiply it with the static gain K. So, this is my general transfer function.

So therefore, what would be $\bar{y}(s)$, would be transfer function multiplied by the Laplace transform of the step input. So, that would be,

$$\bar{y}(s) = Ak\frac{1}{s}\left(\frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_q s^q}{(s - r_1)(s - r_2)\ldots(s - r_p)}\right)$$

How will I do the Laplace inverse inversion? Well, I know that I will need to do partial fraction. So, imagine that I can factorize the denominator, I do not worry about the numerator, again, I need to factorize the denominator. And let us say that the denominator can be factorized as follows. The numerator is simply $a_0 + a_1 s + a_2 s^2 + \ldots + a_q s^q$, and I have factorized my denominator into p number of factors.

So let us say the factors are $(s - r_1)$, $(s - r_2)$ up to $(s - r_p)$, what would be my next step? I will need to do a partial fraction, I will not worry about what exactly would be the form of the partial fraction, but I know what would be the functional form of the final finally partial final partial fraction which I get. I hope you would agree that the final partial fraction would look like this, $\frac{c_0}{s} + \frac{c_1}{s-r_1} + \frac{c_2}{s-r_2} + \ldots + \frac{c_p}{s-r_p}$, there would be some numbers associated with c_0, c_1 and so, on. I do not need to worry about that, at this point of time because they are not going to affect the stability of the system, they would affect the dynamics, but it will not affect the stability.

We will see why, what is going to happen to y(t)? So, now we will do a Laplace inversion. So, I will get $y(t) = AK(c_0 + c_1 e^{r_1 t} + \ldots + c_p e^{r_p t})$. Now you see the first term on the right-hand side is a constant function it is always going to be bound, but rest every term on the right-hand side has $e^{r_i t}$.

So, I have e to the power what is the nature of e^{rt}, this is what this is e^{rt} versus t it looks like this, which means unbound and e^{-rt} and in every case r > 0. So, e^{-rt} you will have e^{-rt} again greater than 0 versus t, this will this is bound. You can find the minimum maximum in that case. So, when r_i is real when r_i is a real, stability dictates that r_i must be less than 0 for y(t) to be bound, if y(t) has to be bound, then our r_i have to be less than 0, you can see the graph on the right-hand side.

Now the problem is when r_i belong to complex numbers, when r_i belong to complex numbers, then what happens? So, I have $e^{r_i t}$ and let me write $e^{r_i t}$, as what $e^{|r|e^{i\theta}}$, where theta is equal to tan inverse imaginary upon real, I can do this. So, then what is going to happen? In this case, even before this happens what I can do is, I can simply write $e^{r_i t} = e^{(z_1 + iz_2)t}$. In this case, I do not need to even convert it to the rθ form, I simply convert it to $r_i = z_1 + iz_2$, so this is what? $e^{z_1 t}e^{-iz_2 t}$.

And now it is very clear that this will give you oscillations to your system because those can be converted to sines and cosines and this will give you decay or growth depending upon z_1. So, therefore, you will have decay for $z_1 < 0$ and growth for $z_1 > 0$, why? Because you have $e^{z_1 t}$, same thing which is there on the figure on the on the right-hand side. So, therefore, it is pretty clear that when you have a system whose poles are all negative or the if the poles are complex numbers, all of the real parts are negative, it is only then that you can expect the system to be stable.

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So let us quickly look into the statement which we had a continuous system is stable if every pole associated with the transfer function has a negative real part or is itself negative. We looked into the transfer function for a first order system and we got further equation

$$\tau \frac{dy}{dt} + y = ku$$

from here we got $g(s) = k / (\tau s + 1)$, what was the response of such a system subjected to $y(t)$ and $u(t)$ as a function of t, if this is A, the response was bound bounded and this was the response. What was the pole? z_r in this case was minus $1 / \tau$ which is less than 0. I encourage you to find this out that for an equation of the form $\frac{dy}{dt} + y = ku$ for which g(s) would be $k / (\tau s + 1)$ your r will become $1 / \tau$ which is greater than 0.

So we had $y(t) = AK(1-e^{t/\tau})$ as the solution. If you follow the exact same procedure here you will find that $y(t) = AK(1-e^{t/\tau})$. And what would this function look like? This function would look like this, this is y(t), u(t) versus t. This is A and what would the response look like when you can plot the response looks like this. So, bounded and tau was minus $1 / \tau < 0$ unbounded was $1 / \tau$ which is greater than 0 as simple as that. So, you can simply look at the transfer function and comment on the stability.

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Now can we extend this to discrete system? Let us look and write down the general discrete general discrete time pulse transfer function. So, g(z) would be what?

$$g(z) = \left(\frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_q z^{-q}}{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_q z^{-q}}\right) \mathrm{K}$$

So, this is the general p, q order discrete time pulse transfer function. We will do the same analysis subject this to step input, what is going to happen? y(z) subject to a step input would be

$$\hat{y}(z) = AK \frac{1}{(1 - z^{-1})} \left(\frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_q z^{-q}}{(1 - r_1 z^{-1})(1 - r_2 z^{-1}) \ldots (1 - r_p z^{-1})}\right)$$

Now let me factorize the denominator. If you remember the lecture in which we introduced z transforms, we can factorize the denominator now into p number of factors as this $1 - r_1 z^{-1}$, $1 - r_2 z^{-1}$, ...+ $1 - r_p z^{-1}$, this is the way we will do the factorization. Then what we will do same, we will do a partial fraction, so what is going to happen? This will be $AK\left(\frac{c_0}{1-z^{-1}} + \frac{c_1}{1-r_1 z^{-1}} + \frac{c_2}{1-r_2 z^{-1}} \ldots + \frac{c_p}{1-r_p z^{-1}}\right)$.

And therefore, y of n t the inverse z transform will give you what? A k times c_0 because one of our z inverse is simply z transform of the step function, the unit step function plus c_1, we will take the inverse Laplace and this would be e to the power nlnr$_1$ please refer to our previous lecture, you will get this plus $c_2 e^{n \ln r_2}$ plus upto $c_2 e^{n \ln r_p}$. So, the first term on the right hand side is bounded with c 0 is a constant.

Now every term rest of the of the terms have $c_2 e^{n \ln r_i}$ in general r_i may be complex. So, therefore, let me write this as $e^{n \ln |r| e^{i\theta}}$. And therefore, this would result in $e^{ni\theta} e^{\ln |r|}$. So, now what we see here is that we have $e^{\ln r}$, and what is ln r? $|r| = \sqrt{z_1^2 + z_2^2}$.

So, if I draw a plane in which I have z_1 the real part on x axis and z_2 the imaginary part on the y axis, then I have a unit circle and then for ln r < 0, you need | r | < 1, the log of quantity less than 1 is negative. So, therefore, in this region as long as you have z_1, z_2 in this region your system is stable otherwise, the system is unstable. And I can draw a corresponding plot, so this is for discrete and what will be the corresponding plot for continuous you have z_1, you have z_2, all of this has the real part which is less than 0, so therefore, you have a system which is stable here.

So you can always draw the complex plane and locate the poles there for the case of continuous system, the poles lie on the second and third quadrant the system would be stable. For the case of a continuous system, you draw a circle of unit length and if the pole lies inside it or even on it, you will have stability otherwise you will not have stability.

So, this is not this is what are what we have we had to study we which we had planned on studying, we started with the analysis of the systems in transform domain studied everything about the continuous domain analysis then we went to the draw the discrete time domain analysis. We studied the effect of the form of transfer functions on the dynamical features, the speed of dynamics basically, and today we also looked into the effect of the features of the transfer function on the stability of the system.

In the next lecture which will be the final lecture we will recapitulate everything which we studied in this particular course. And for today, we will stop here, we meet again for the last lecture tomorrow. Till then, goodbye.