Advanced Process Dynamics
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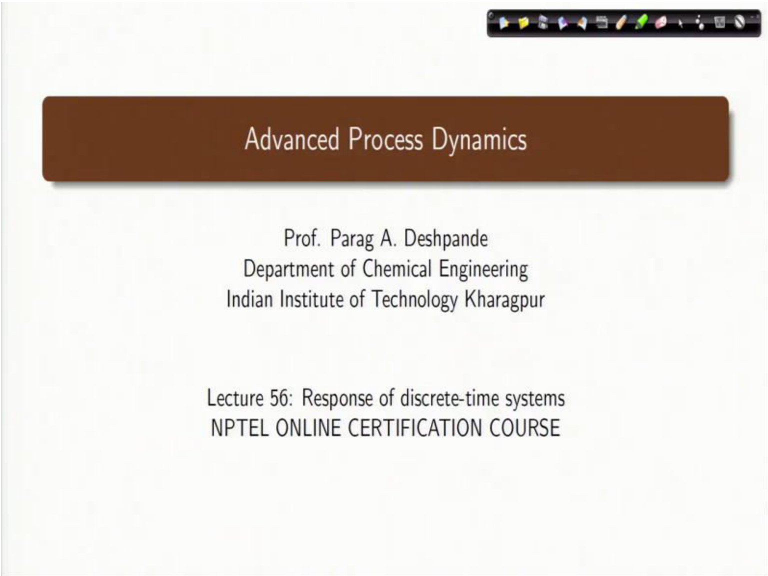
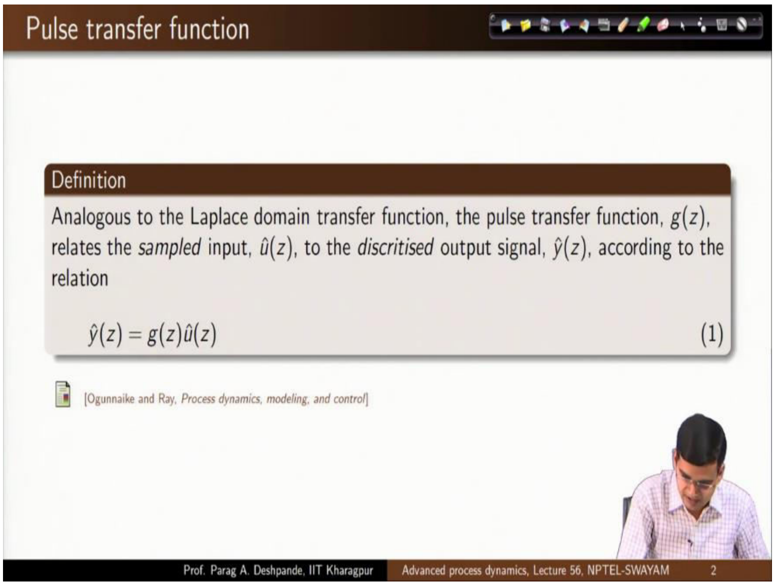
Lecture 56
Response of discrete-time systems





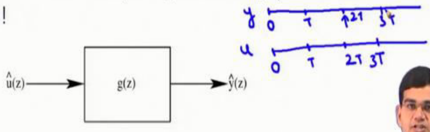
Pulse transfer function

Definition

Analogous to the Laplace domain transfer function, the pulse transfer function, $g(z)$, relates the *sampled* input, $\hat{u}(z)$, to the *discritised* output signal, $\hat{y}(z)$, according to the relation

$$\hat{y}(z) = g(z)\hat{u}(z) \tag{2}$$

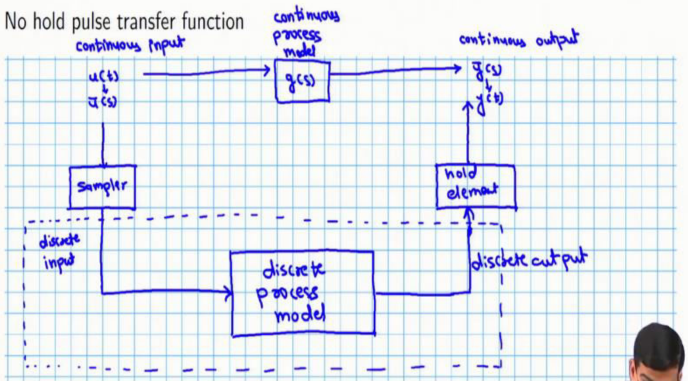
The input and output signals must be sampled synchronously (i.e., at the same time), and also at the same rate!!!



[Ogunnaike and Ray, *Process dynamics, modeling, and control*]

Pulse transfer function

No hold pulse transfer function



Pulse transfer function

No hold pulse transfer function

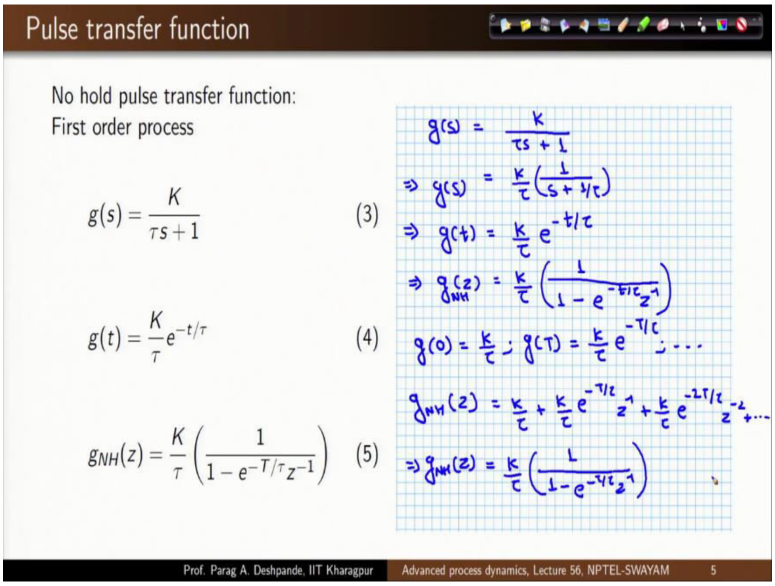
Laplace domain transfer function – $g(s)$

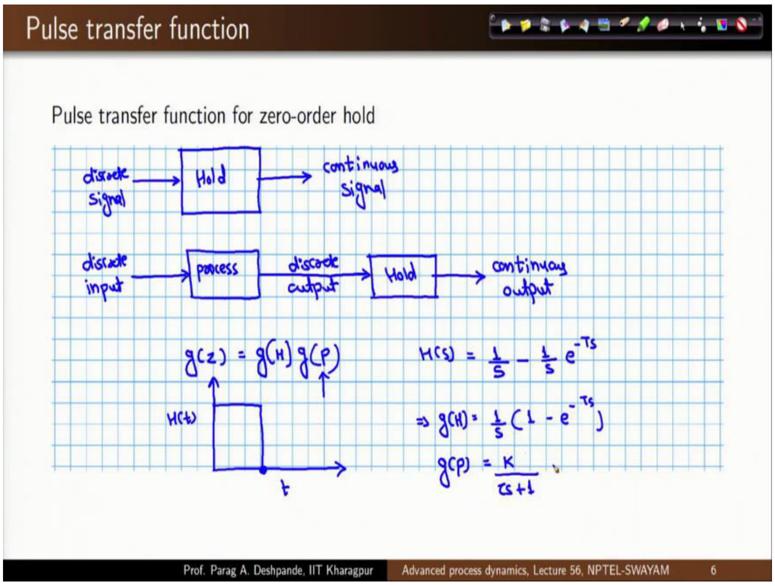
No hold pulse transfer function – $g(z)$

$g(s) \longrightarrow g(t) \leftarrow$ continuous function of t

$\downarrow$ sampling

$g(z) \longleftarrow g(T), g(2T), g(3T) \cdots$





Hello, and welcome to this last week of instruction on this online course on advanced process dynamics. We are currently studying discrete-time systems the dynamical response of discrete time systems.

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And in the previous week, we develop the motivation behind that requirement of discrete time systems, which becomes inherently applicable when you use digital computers for your control, how the discrete input is generated, what is the requirement of conversion of models to discrete time models and how is a discrete output from the models converted to continuous output.

We also introduced you to a new mathematical concept called z transformation similar to the Laplace transformation in continuous domains z transformation is expected to help us in understanding the dynamical response of discrete time systems. So, let us look into this particular aspect of response of discrete time systems.

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So, what we have in front of us is something called a pulse transfer function. So, before we define what a pulse transfer function is and how does it differ from transfer function in Laplace transform domain. Let us try to understand the larger scheme. So, the scheme that we have in front of us is that we have a situation where the input output model equation is available with us.

So, typically, this would be a system in which you have dynamical equation or set of dynamical equations, if it is a higher order system and you would have an output equation if it is a single output system or a set of output equations if it is a multiple output system. So, in general, you may have a multiple input, multiple output system following which you may have a system of dynamical equations and a system of output equations.

Now, this dynamical equations as well as output equations would be in terms(t)ime. So, in state space domain, what you will do is you would like to do the analysis(t)he phase portraits if you are not in a position to explicitly solve for the dynamical variable output equation. But otherwise also you would like to solve the equation and try to determine how does the dynamical variable and the output equation output variable change with time.

We took a special situation where we were interested in the response(t)he system rather than every possible state that the system can attain. We were interested in the response(t)he system and that was obtained in using the Laplace transform of the equations. So, rather than dealing with t directly, we want transformed our system from t domain to s domain via Laplace transformation.

Now, our intention was to determine the dynamical response(t)he system subjected to some forcing function, which you call the input function. So, if I have the input function, I should in principle be in a position to solve for x(t) and y(t). And in this particular case, now, I have $\frac{\bar{y}(s)}{\bar{u}(s)}$ and which is referred to as the transfer function of your system, $\frac{\bar{y}(s)}{\bar{u}(s)}$.

So, the scheme is that from the model equations, you determine the transfer function. How would you do that? You will take the Laplace transform of the dynamical variable from where you will get $\frac{\bar{x}(s)}{\bar{u}(s)}$. The Laplace transform of the dynamical variable in deviation variable form divided by the Laplace transform of the input variable in deviation variable form.

Similarly, you will do the Laplace transform of the output equation from where you will get after several rearrangements and simplification, $\frac{\bar{y}(s)}{\bar{u}(s)}$which means the Laplace transform of the output variable divided by the Laplace transform of the input variable. What do I require? I require the time variation of my output from the system. And therefore, what I can do is given that I have g(s) which is $\frac{\bar{y}(s)}{\bar{x}(s)}$ I will identify the input function. I will take its Laplace transform. So, I will get $\bar{u}(s)$.

So, if I multiply g(s) by $\bar{u}(s)$, I will get $\bar{y}$(s) which means I will have the Laplace transform of the output variable and I can do an inversion. So, inverse Laplace transform of $\bar{y}(s)$will give me simply the time evolution of my system. So, this is the larger scheme which we followed for the analysis of continuous domain systems.

Now, what we have done is that we have converted our continuous system to a discrete time system. So, again, the continuous input has been converted to a corresponding discrete time input. My continuous model has been converted to the corresponding discrete time model. And as a result from discrete input and discrete model, I will get a discrete output.

So, therefore, in principle, I can define an analogous quantity, which takes in the discrete input that operates on the equivalent quantity, which is a quantity which is equivalent to the Laplace transform in continuous domain, it would be operated by the discrete time input and it would give you the output, discrete output or continuous output at this point(t)ime, I am not saying anything about that, all I am saying is that this quantity will give you the output.

Now, the input is a discrete signal. So, therefore, instead of Laplace transform of the signal you will have the z transform of the signal. Similarly, when you described transfer function in Laplace transform domain instead of g(s). Now, since the entire system is in discrete time domain, you will have a quantity g(z) because that whatever that transfer function is, would be a functional z. And that is precisely what is the meaning of pulse transfer function. So, let us see what is the meaning of pulse transfer function. It is the function *g(z)* that relates the sampled input.

Now, this is important that *g(z)* takes the input as the sampled input. So, your process may be continuous, but inevitably, you will have to convert that continuous input to a discrete input. So, the input to the pulse transfer function would be the sampled input which means it would be a discrete input to the discretized output, the output also would be discretized. And that will follow the relation output the z transform of the output $\hat{y}(z)$ would be equal to the pulse transfer function g(z) multiplied by the sample input z transform $\hat{u}(z)$. So, equation one gives you the definition of purse transfer function $\hat{y}$ (z) = g(z) u (z).

Now, one thing which is important is that you have added the term pulse transfer function. It in itself gives you an idea that you are dealing in discrete time domain. You are no more in continuous domain. So, therefore, you do not use to emphasize that you are sampling a discrete signal and giving output also as a discrete signal by processing using discretized models, you use the term pulse transfer function and what is the reason behind use(t)he term pulse because we said that how would you sample the input you will sample the input using a sampler there will be a switch and therefore, in a sense, your continuous signal would be in the form of pulse, ideally it should be an impulse in that case, where delta t tends to 0, but realistically, it would be a pulse.

So, we are in general using the term pulse transfer function. So, it is important to realize that we have transfer function g(s) for continuous domain analysis and we have a pulse transfer function g(z) in the discrete time domain analysis and the transfer function will involve in this particular case, the z transform rather than the Laplace transform. Let us see if we can make any more comments on this.

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So, now, when you have a transfer function in continuous domain g(s), the basic utility of this entire exercise was that you could develop a simple input output model. So, you could diagrammatically represent the system using a block diagram, where you have a single input as you can see, in this diagram, you have a single input single output, input being u, the output being y. Now, there used to be a box. Box basically means a set of mathematical operations and in the previous case, $\bar{u}$(s) as an input to the box would give $\bar{y}$(s) and the box would have the transfer function g(s).

Now in an exact analogous manner, you can make a block diagram for pulse transfer function as well. So, what is the meaning of pulse transfer function? Well, if you apply the pulse transfer function to the input, you will get output. So, you can draw a block diagram which looks like this. You have u hat of z which acts as the input to the box, the box has mathematical operations, that particular operation in this case is the z transformation of the input output model discretize model that you have and the output that you will get would be $\hat{y}(z)$. But, in case of Laplace transformation, well you had a continuous time input, you had a continuous time output.

So, therefore, there was not much of an issue regarding the location at which you are determining your functions, because it is a continuous system, this function was defined at all locations of time or all instances of time. But in this case, we actually have in the previous lecture, seen the importance of the sampling, all of your analysis depends upon the sampling time. And we also said that the z transform is not unique, it depends upon the location. So therefore, if you change the sampling interval, there is a transform will change.

Therefore, it is important that when you define a pulse transfer function, you have a statement here on the screen, the input and output signals must be sampled synchronously, which means that if you are having, so let me do it diagrammatically that, if this is u for input and you are doing it sampling it here at 0, T, 2T, 3T and so on. Your output will also be at 0, T, 2T, 3T and so on the same time and also at the same rate, of course, this goes without saying that, if the input is at a time interval T, capital T, the output also should be at capital time interval T. So, this is the general idea about the pulse transfer function.

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So, let us see if we can have something about the transfer function pulse transfer function in a little greater detail. So, the first thing which you would like to know is the no hold transfer function. Let us try to remind ourselves of the scheme that we are following. So in a continuous domain analysis, you have u(t) from where you get $\bar{u}$(s), you do the Laplace transform. And then what you do is you feed it to g (s), the transfer function when you apply g(s) to u(s), what you get is $\bar{y}$(s) from where you get y(t), very simple.

So, you have a continuous input and you get a continuous output simple. Now, you implement a digital control system, so, it is imperative to convert this entire business to discrete time domain and how does this scheme change now? Your continuous input will pass through a sampler; we make use of the analogy with a switching element. So, you will pass it to the sampler. And what would be the output from the sampler? The output from the sampler would be a discrete input. Now, what you would do is you would feed this discrete input to another box, which comes as the transformation we do not know the notation for it, but it would do the transformation(t)he discretized model.

So, this is the process model and in fact continuous process model. Now, I am feeding my input to a discrete process model. So, I will have to do this discrete process model. Well, I will feed this discrete input to my discrete process model and if I am printing it to my discrete process model, obviously, I will get a discrete output and then what happens, the discrete output is sent to a hold element if you remember what we studied in the previous lecture. This is sent to a hold element and the hold element would convert these discrete outputs to a continuous output and this is what you get. So, this is the larger scheme.

And now, what we have done here is that we have a no hold situation, no hold pulse transfer function which means what I send in the continuous signal. It is processed by the model but my output is still discrete. So, therefore, I have this situation with me. So, the input is discrete. Well input has always got to be discrete because you are doing z transformation and the only input which z transformation will work on is discrete input. So, the input is discrete, but the output is no more continuous. In fact, you have not allowed the output to be continuous at all. And therefore, your output also is discrete and it goes through some mathematical jugglery with in this box and that mathematical jugglery would be called the no hold pulse transfer function.

So, now the question is how would you determine the no hold transfer function? Well, it is very simple.

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The way to determine the no hold transfer function is this that you will have a rather than continuously let us write Laplace domain transfer function. Let us imagine that you have a Laplace domain transfer function, which is available to you. And that is g(s), you would like to determine the no hold transfer function for this system. Assuming that you can convert this process model to a discrete model that is a step which we have already learned and that should not be an issue at this point of time for this analysis. So, how would I determine no hold pulse transfer function and let me denote that by g of z at this point of time. So, I need to do this from g of s, I will obtain g(t).

Now, this is strange, because at no point of time till today, we have done the inversion(t)he transfer function itself, because we were interested in determining the dynamical response(t)he system. So, in continuous domain, we used to multiply it with the Laplace transform of the input function. Here, we are not doing that we are converting g(t). So, therefore, g(t) is a continuous function of t. So, therefore, what I would do is now, I would sample, so, I would do sampling. So, what I would get? I would get g at T, g at 2T, g at 3T and so on.

Now, this would act as the input to my z transform and therefore, from here using the usual definition I can determine g of z and this would be the no hold pulse transfer function. Let me repeat the procedure, I started with g(s), I inverted it determine various points, the value of the continuous function g(t) at various sampling points and then use those for determining g(z).

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Let us take an example and see if we get this. So, let us say I have a first order process and I want to determine the no hold pulse transfer function. So, I have g(s) = k / (τs + 1). So, this is equal to $\frac{k}{\tau(s+1/\tau)}$, this is g(s) from where I can get g(t) = $\frac{k}{\tau}e^{-t/\tau}$

Now, you can as well use this continuous function directly to write from here, g(z) and to emphasize that this is a no hold pulse transfer function. I am writing NH is equal to what, $\frac{k}{\tau(1-e^{-t/\tau}z^{-1})}$ We have written directly. You can as well do one thing you can write g(0) = k/τ, g(t) = (k / τ) ($e^{-T/\tau}$) and so on. So, these are the very sampled inputs for your z transform. So, therefore, I can write g no hold z is equal to what, the formula was summation from n is equal to 0 to infinity.

So, therefore,

$$g_{NH}(z) = \frac{k}{\tau} + \frac{k}{\tau}e^{-T/\tau}z^{-1} + \frac{k}{\tau}e^{-2T/\tau}z^{-2} + ..$$

from where I get the same answer using because that is the ratio of two subsequent terms in the series the progression. So, this is how you determine the no hold transfer function.

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Now, you have in general a hold because you would like to ultimately get the output as a continuous function. So, instead of analyzing the output only as a discrete I am now implementing a hold and what I will do is I will using the hold I will convert that discrete signal to a continuous signal. So, what does the hold element do? So, you have a discrete signal coming from the process. It is fed to your hold element and you get a continuous signal.

Now, what you need to do is this that how do you get the discrete signal? Well, you have the discrete input which would always be the case because you are dealing with z transformation. It goes through the process, the process generates discrete output and this is fed to a hold element and hold element gives you continuous output. So, now, if I need to define a transfer function between the continuous output and the discrete input, I have two boxes here, I have two blocks and using the usual principles, I know that the overall transfer function would be simple multiplication(t)he individual transfer function blocks.

So, therefore, my pulse transfer function for zero order hold. So, g(z) the pulse transfer function for a zero order hold, will have what? Will have the transfer function for the hold multiplied by the transfer function for the process, the transfer function for the hold multiplied by the transfer function for the process. And what is that which I now need to know transfer function(t)he process will come from your discrete model.

For example, in the previous case we looked into the first order process and then we try to convert it to a discrete so, that g of p will be known to you, what you need to know is you need to know g of h that the transfer function corresponding to hold and what does the hold do? The hold does is this so, this is h(t) versus t the hold keeps your output to that constant value and then puts it down to zero that is the hold. So, for this particular process, what would be h(s)? The h of s would be for this pulse for the rectangular pulse which is $\frac{1}{s} - \frac{1}{s} e^{-T/s}$. In other words,

$$h(s) = \frac{1}{s}(1 - e^{-T/s})$$

So, this is the Laplace transform of the hold element and the Laplace transform of the first order system for example, like the previous one is this, so, this is h(s) and then g(s) =K/(τ s+1). So, this is g(s) and let us be consistent with the notation let me write this as g(H), Laplace transform the hold and the Laplace transform of the process. So, g(p) for the process. So, what would you do to remind the zero order pulse transfer function? Will do g(H) multiplied g(p) and repeat the same procedure as you did previously.

In the interest of time, we will stop here today and continue this discussion of pulse function for zero order hold in the next lecture and also see the dynamical response of a discrete time system given that you have the pulse zero order function hold function available to you. Till then, goodbye.