Advanced Process Dynamics

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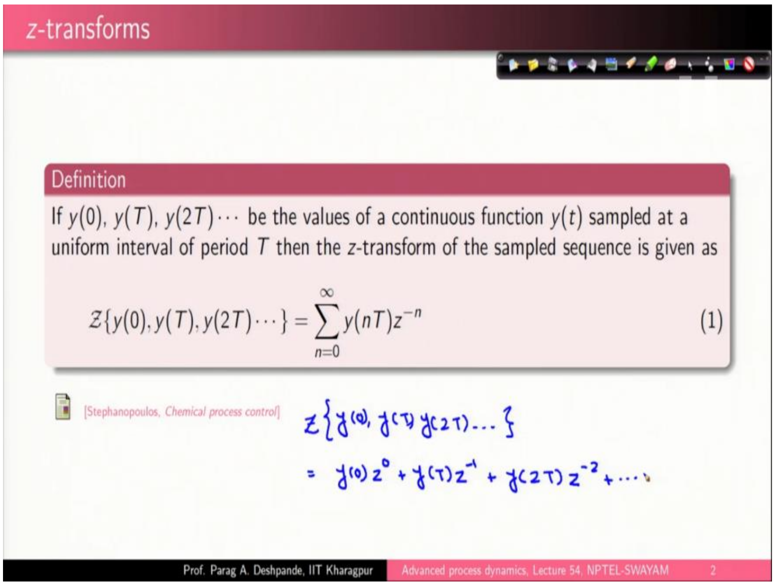
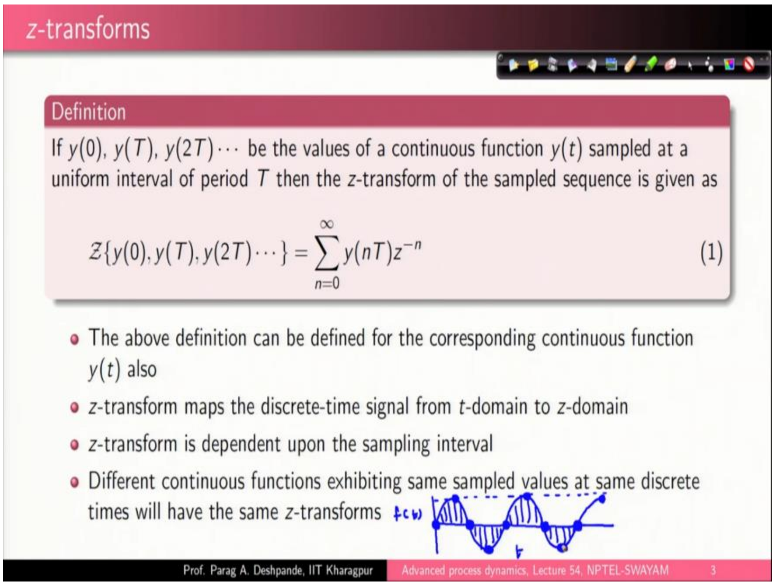
Lecture 54

Introduction to z-transforms





z-transforms

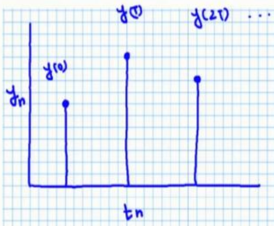
z-transform is a special case of Laplace transform

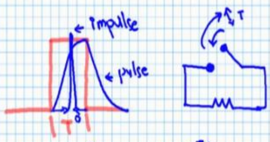
$$y(t) = y(0) + y(T) + y(2T) \cdots \quad (2)$$

$$y(t) = y(0)\delta(t) + y(T)\delta(t-T)$$
$$+ y(2T)\delta(t-2T) + \cdots \quad (3)$$

$$y(t) = \sum_{n=0}^{\infty} y(nT)\delta(t-nT) \quad (4)$$

$$\bar{y}(s) = \sum_{n=0}^{\infty} y(nT)e^{-nTs} \quad (5)$$





$$\bar{y}(s) = \sum_{n=0}^{\infty} y(nT)e^{-nTs}$$
$$\hat{y}(z) = \sum_{n=0}^{\infty} y(nT)z^{-n}$$
$$e^{-nTs} \to z^{-n}$$

Some z-transforms

z-transform of a constant function

$$y(t) = c \quad (6)$$

$$\hat{y}(z) = c\left(\frac{1}{1-z^{-1}}\right) \quad (7)$$

$y(t) = c$ —(1)

$\hat{y}\{y(0), y(T), y(2T) \ldots\}$ —(2)

$= \sum_{n=0}^{\infty} y(nT)z^{-n}$

$y(0) = c;\ y(T) = c;\ y(2T) = c \ldots$

$\Rightarrow \hat{y}(z) = \sum_{n=0}^{\infty}\{c + cz^{-1} + cz^{-2} + \ldots\}$

$= c\sum_{n=0}^{\infty}\{1 + z^{-1} + z^{-2} + \ldots\}$

$= c\left(\frac{1}{1-z^{-1}}\right)$

Some z-transforms

z-transform of exponential function

$$y(t) = e^{-at} \quad (8)$$

$$\hat{y}(z) = \left(\frac{z}{z - e^{-aT}}\right) \quad (9)$$

$y(t) = e^{-at}$

$\hat{y}(z) = \sum_{n=0}^{\infty} y(nT)z^{-n}$

$y(0) = 1;\ y(T) = e^{-aT};\ y(2T) = e^{-2aT} \ldots$

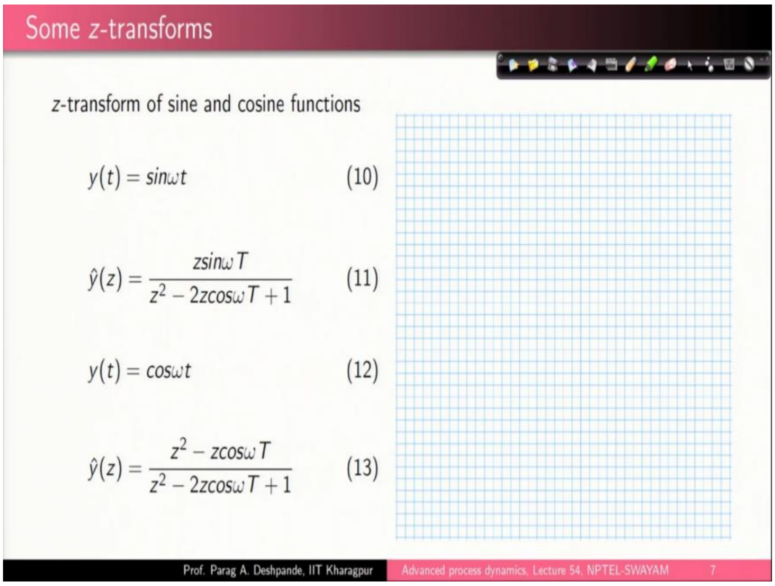
$\hat{y}(z) = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + e^{-3aT}z^{-3} + \ldots$

$r = \frac{T_{n+1}}{T_n} = \frac{e^{-2aT}z^{-2}}{e^{-aT}z^{-1}} = e^{-aT}z^{-1}$

$= \frac{e^{-3aT}z^{-3}}{e^{-2aT}z^{-2}} = e^{-aT}z^{-1}$

$\Rightarrow \hat{y}(z) = \frac{1}{1 - e^{-aT}z^{-1}}$

$\Rightarrow \hat{y}(z) = \frac{z}{z - e^{-aT}}$



Welcome back. We are analyzing the dynamics of discrete time systems and as I made a mention in the previous lecture for understanding the dynamical response of discrete time systems, we will need a special technique called z-transforms. I assumed that when you took up the lecture on dynamical response of continuous systems, we made use of Laplace transformations and you already knew about Laplace transformations.

But z-transformations, something very new to you. I assume that you might not have come across this particular concept and therefore before trying to understand the use of z-transformations for understanding the response of dynamical systems in discrete time domain, let us look into the underlying mathematical theory. So, we will spend some time on the mathematical theory of z-transformations. So, let us see what exactly z-transformations are.

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We would be using z-transformations for the analysis of discrete time domain systems. So, what is the definition of z-transformation? So, if y(0), y(T), y(2T) and so on are the values of a continuous function y(t). So, our general scheme is that you have a continuous function. Your dynamical variable is continuous but since this continuous variable cannot be fed or may not be fed to a digital computer, you would be sampling the continuous variable at instances of time 0, T, 2T, nT and so on.

So, you have only these discrete values available with you. So, if y(0), y(T), y(2T) and so on are the sample values of a continuous function y(t), which is sampled at uniform interval of time T. Then the z-transform of this sampled sequence, that transform of this sampled sequence is given by this equation (1) and let us first see the left-hand side, the notation is z for z-transformation, within the bracket you have the sequence.

Remember that you do a Laplace transform of a function. So, you have L of a function. L(f(t)) for example. Now, you do not have a function, you have discrete values. So, those discrete values have been listed in the bracket and this is equal to a summation. The Laplace transform was an integral from 0 to ∞.

Here since, you are dealing in discrete time domain, you have a summation here. You have a series and the series is given as

$$\mathcal{Z}\{y(0), y(T), y(2T) \ldots\} = \sum_{n=0}^{\infty} y(nT)z^{-n} \ldots \ldots \ldots (1)$$

So, what would this series look like? This would look like.

So,

$$\mathcal{Z}\{y(0), y(T), y(2T) \ldots\} = y(0)z^0 + y(T)z^{-1} + y(2T)z^{-2} + \cdots$$

This infinite series is your z-transformation. And obviously, this series must be a convergent series for your z-transformation to exist, that goes without saying. So, now, before we make use of any of the concepts associated with z-transformation, let us look into some of the properties.

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So, the definition is in front of you. Now, what we have is that the z-transformation is applied to a sequence of discrete numbers. So, therefore you have $\mathcal{Z}\{y(0), y(T), y(2T) \ldots\}$, but these numbers must be coming from some function. As we made a mention these are the sample values at different instance of time and that continuous variable must be following some function. It must be some function of t.

So, therefore the z-transformation in principle can be defined for the corresponding continuous function also, continuous function y(t) also. So, you can define it. However, your transformation would be defined only at those instances of time, not in the entire continuous-domain.

So, you can say that I have the z-transformation of these points or I have the z-transformation of the function at these locations of the independent variable. Both are allowed. But in essence, you are go, whether you do in discrete time or you do the z-transformation of a continuous function, you are doing it at discrete instances of time only.

Then z-transform maps the discrete time signal from T-domain to z-domain. Well, that is the meaning of transform anyway. In case of Laplace transform, you have a function which is a function of t and when you do a Laplace transform, you get a function of s. So, your Laplace transform maps from t-domain to s-domain.

Here the definition goes as $\mathcal{Z}\{y(0), y(T), y(2T) \ldots\} = \sum_{n=0}^{\infty} y(nT)z^{-n}$. So, therefore your z-transformation is mapping your discrete time signals from T-domain to z-domain. And z-transform is dependent upon the sampling interval. This is very important. Because you have your signals at certain locations only.

If the sampling interval changes, the intensities of the variables at those locations change and therefore it is important to realize that z-transform is meant for those quantities which have been given in the brackets of equation number (1). So, if these quantities change, your z-transform may change.

And this gives rise to the importance of sampling. The importance of number of samples that you have generated out of your system. Because the z-transform may be different for the different samples which you have generated. And finally, different continuous functions exhibiting the same sampled value at the same time will have same z-transform. Because all what a z-transform is doing is taking the values from the signals.

Now, imagine that you have a sinusoidal signal which is a periodic signal and you start sampling only when the signal is at its maxima. So, let us look into this particular aspect. I have a signal which goes like this. This is f(t) versus t. So, what I should ideally do to capture the correct characteristics of this function?

I should capture the signals at very small instances of time, so that what you see as vertical lines is what you get. This is what ideally should happen and therefore you would have the values of the signal corresponding to all of these vertical lines. Now, imagine that instead of this, I do sampling such that I have only samples here. Then what would happen?

Well, I would have the signal which looks like this, a straight line and this would also correspond to a constant signal of amplitude which is same as the maxima of your sine function. If my interval is like this that I have this these values with me, looks like you will get a wrong impression that the value of the intensity of signal is 0. And the third case as well that you have all negative values and so on.

So, therefore, depending upon, the signals depending upon the time of interval in which you do your sampling, your signal may be different. And then you see that you are getting same values for discrete signals for same sinusoidal function and none of them are capturing the correct characteristics of your system. Therefore, different continuous functions having the same value of the function at those locations will have the same response transformation and therefore you need to be careful on how to sample the signal, how to convert your continuous signal to discrete time signals.

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If this be the case, can I establish a correspondence between Laplace transform and z-transform? Because I know how to deal with Laplace transform well. I assume that how to deal with Laplace transform well. So, Laplace transform takes you from t-domain to s-domain. z-transform takes you from T-domain to z-domain.

But whether you are dealing in with continuous systems or discrete systems, the system remains the same; so therefore, the characteristics which you should in principle be in a position to extract should also remain the same. And therefore, the framework as a whole should not differ very much.

So, now, in the one of the previous lectures, we said that what I can do is this that I have a signal y_n........t_n which has intensities like this. Let us assume that the signal has intensities which are like this. So, this is y(0), y(T), y(2T) and so on. But how do I practically get this? I get this because I have a switch and I switch on the switch and I switch off the switch in the time interval T.

And the dynamics of every element associated with this process will may have some time constant. So, therefore, there may be some inherent delay in the dynamics of your system and therefore ideally what you do is you must have something like this. Where this is T. If you, when you switch close the switch on and then put it off, put the switch on and close it off that interval is T and therefore within that T what actually happens is that you rise and then you fall.

What you would like, what you would like is to tend this to 0. So that you rise and come back as an impulse rather than a pulse. This is what you would like. And the area under the curve is, in fact, the value of the signal y(0), y(T), y(2T) and so on which you are observing here. If that is the case then your signal is given as

$$y(t) = y(0) + \ y(T) + y(2T) \dots \dots \dots \dots (2)$$

And since you are approximating this as an impulse function, what you have in your, as your equation number (3) is

$$y(t) = y(0)\delta(t) + \ y(T)\delta(t-T) + y(2T)\delta(t-2T) \dots \dots \dots \dots (3)$$

which can be written in summation as equation number (4)

$$y(t) = \sum_{n=0}^{\infty} y(nT)\delta(t-nT) \text{..........}(4)$$

and then you use the shifting property of Laplace transform to get

$$\bar{y}(s) = \ \sum_{n=0}^{\infty} y(nT)e^{-nTs} \text{...........}(5)$$

And what is the formula for z-transform? So, the notation for z-transform would be a hat (^) over the function. So,

$$\hat{y}(z) = \sum_{n=0}^{\infty} y(nT)z^{-n}$$

So, it is not very difficult to see that you can get z-transform from Laplace transform by doing a simple substitution where e^{-nTs} has been represented by z^{-n}. So, there is a one-to-one correspondence between the Laplace transform and the z-transformation.

So, a lot of properties which you see, you have studied for Laplace transformation would hold true for z-transformation as well with the modification which has been shown here. But the general framework which we have used for Laplace transform that if you have a differential equation, you will take the Laplace transformation, do rearrangement, take an inverse Laplace transformation to solve the differential equation, the dynamical equation.

Perhaps that framework can be used in this particular case as well, which gives us a hope that we would be able to use the z-transformation to determine the dynamical response of a discrete time system. So, why are we doing, why are we studying z-transformation at all? We studied Laplace transformation to understand the dynamical response of continuous variable systems.

We have shown here that there is a one-to-one correspondence between Laplace transform and z-transformation. And therefore, we can say that we can study the dynamical response of discrete time systems using z-transformation. How will we do that? We are going to do this over next few lectures.

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So, let us move ahead and see how we can do z-transformation. So, we studied the Laplace transformation of a constant function, did the integration, got the answer as constant function upon s as the Laplace transform of a constant function. Can we do the z-transform of a constant function? So, now, we have a constant function in front of us.

$$y(t) = c \dots \dots \dots (1)$$

And now what we would like to do is to determine its z-transform. And I know that

$$\hat{y}\ \{y(0), y(T), y(2T) \dots\} = \sum_{n=0}^{\infty} y(nT) z^{-n} \dots \dots \dots (2)$$

This is the definition of z-transform. So, let me call this equation (1), let me call this equation (2).

So, for equation number (1) when y(t) =c,

$$y(0) = y(T) = y(2T) = c \ldots$$

The value is constant at every sampling point. So, therefore, the z-transform would be what?

$$\hat{y}(z) = \sum_{n=0}^{\infty} c + cz^{-1} + cz^{-2} + \cdots$$

Which means

$$\hat{y}(z) = c\sum_{n=0}^{\infty} 1 + z^{-1} + z^{-2} + \cdots$$

And what kind of series is this? This is an infinite geometric progression and we know the summation of the series. This would be simply what?

$$\hat{y}(z) = c\left(\frac{1}{1 - z^{-1}}\right)$$

We know the formula for the summation and that is how you get the z-transformation. So, the z-transformation for a constant function c is simply $c\left(\frac{1}{1-z^{-1}}\right)$. Let us see if we can see any other example.

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Now, I have

$$y(t) = e^{-at}$$

I need to know $\hat{y}(z)$.

$$\hat{y}(z) = \sum_{n=0}^{\infty} y(nT) z^{-n}$$

So, let me first determine y(nT) at various locations. So,

$$y(0) = 1; \; y(T) = e^{-aT}; \; y(2T) = e^{-2aT} \; ....$$

So, therefore, I will have

$$\hat{y}(z) = 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + \cdots$$

So, now, I do see a series but I need to make sure whether I really have a geometric series here. So, I can define a ratio ‘r’ as

$$r = \frac{T_{n+1}}{T_n}$$

and what would this be equal to? This would be

$$r = \frac{e^{-2aT} z^{-2}}{e^{-aT} z^{-1}} = e^{-aT} z^{-1}$$

Let me check for one more.

$$r = \frac{e^{-3aT} z^{-3}}{e^{-2aT} z^{-2}} = e^{-aT} z^{-1}$$

Which means I can write this as

$$\hat{y}(z) = \frac{1}{1 - e^{-aT} z^{-1}}$$

which can further be simplified as

$$\hat{y}(z) = \frac{z}{z - e^{-aT}}$$

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Finally, you can as well write in the similar manner, the expression for the z-transform of sin and cosine function. In the interest of time, I will just point out the steps which you would do. Well, you have

$$y(t) = \sin \omega t$$

Again, you would like to write this in terms of the series which is the z-transformation except that now looks like you may not get a geometric series.

But can you convert this to a geometric series? Well, you can. You use Euler's identity.

$$e^{i\theta} = \cos \theta + i\, sin\theta$$

$$e^{-i\theta} = \cos \theta - i\, sin\theta$$

from where you can get the expression for $sin\theta$ and $cos\theta$ in terms of e^{iT} and e^{-iT}.

And then you will see that when you do that, you have the functions which have geometric progression in it. So, when you do this elementary mathematical step, you will get the expression

$$\hat{y}(z) = \frac{\mathrm{z}\sin \omega T}{z^2 - 2zcos\omega T + 1}$$

And

$$y(t) = \cos \omega t\ ;\ \hat{y}(z) = \frac{\mathrm{z}^2 - \mathrm{z}\cos \omega T}{z^2 - 2zcos\omega T + 1}$$

I leave this as an exercise for you to do these elementary operations and ensure yourself that these are the steps; these are the answers that you get. So, we learnt about z-transformations in this lecture.

We will learn the definition and we took some elementary functions to understand how we can perform z-transformation, the way one makes a complete table out of Laplace transformation and have it ready for has it ready for the quick reference for solving the equations, differential equations. One may as well develop a table for z-transformation for a ready reference and use it for inversion of the transformation for solution of problems. We will look at this particular method in the next lecture. Till then good-bye.