Advanced Process Dynamics

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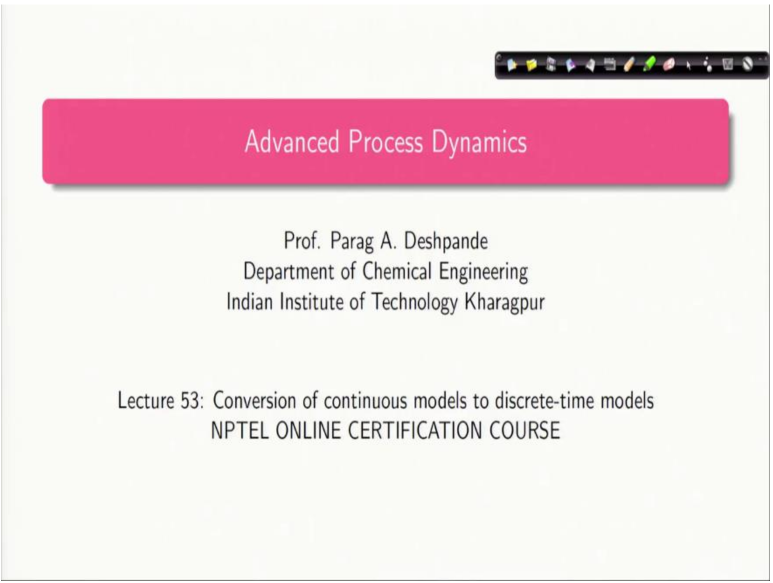
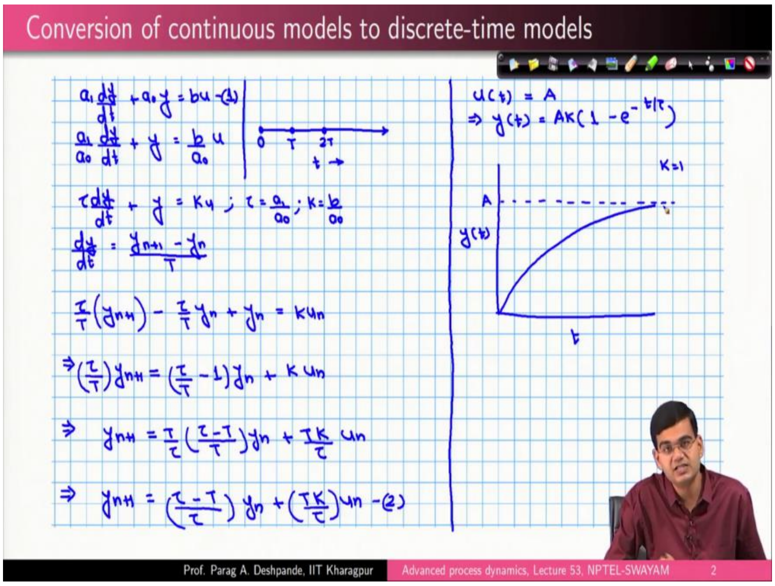
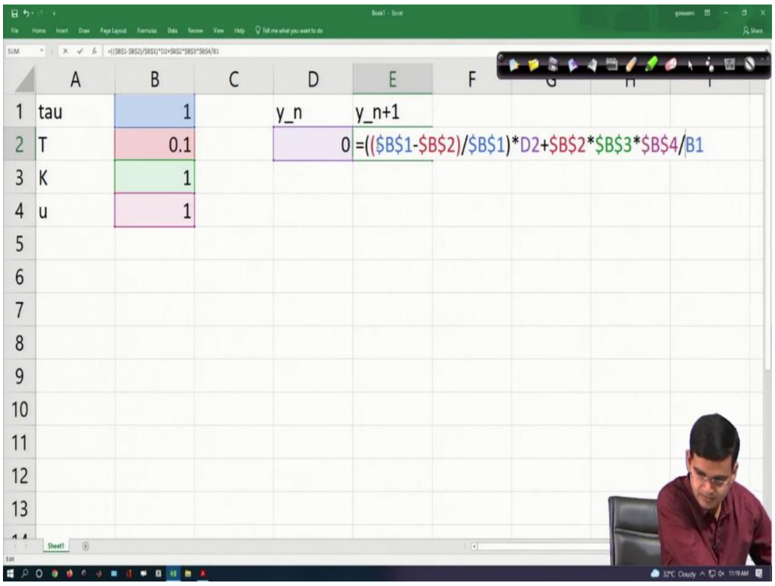
Indian Institute of Technology, Kharagpur

Lecture 53

Conversion of continuous models to discrete-time models



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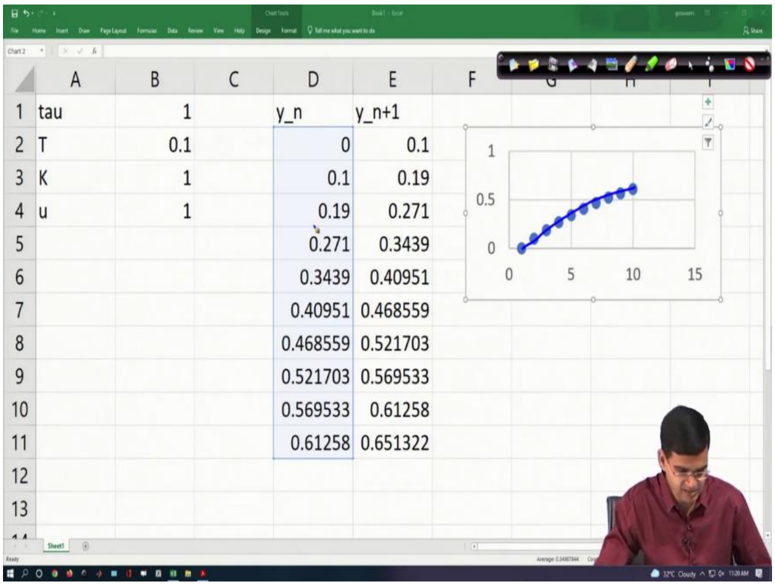
T 0.1
K 1
u 1
y_n
y_n+1
0
=(($B$1-$B$2)/$B$1)*D2+$B$2*$B$3*$B$4/B1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T 0.1
K 1
u 1
y_n y_n+1
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0.40951 0.468559
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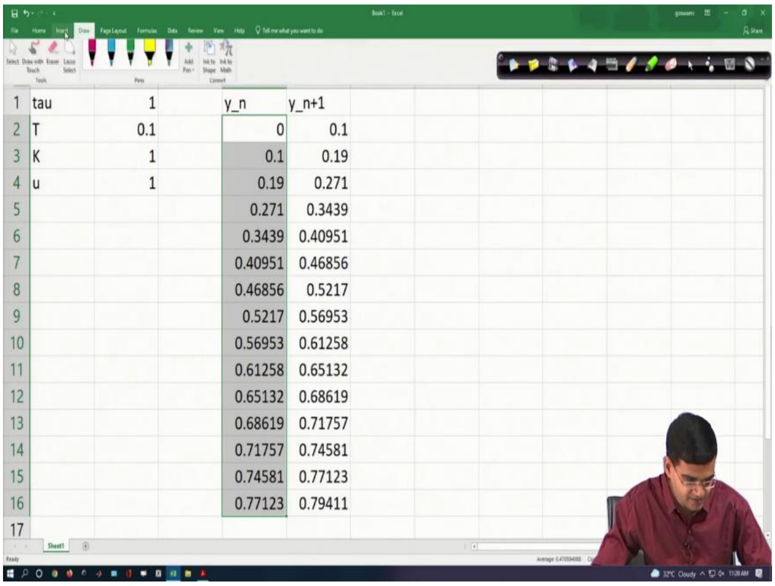
1
0.5
0
0 5 10 15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0.46856 0.5217
0.5217 0.56953
0.56953 0.61258
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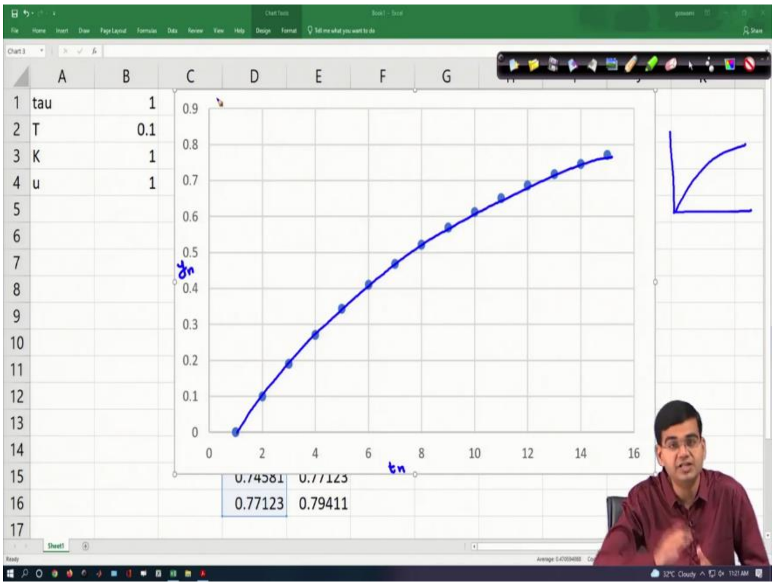
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K 1
u 1
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tn
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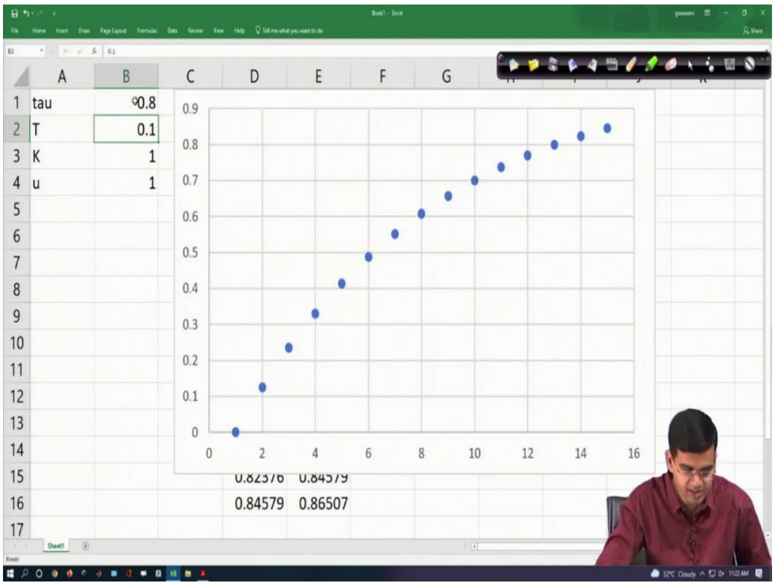
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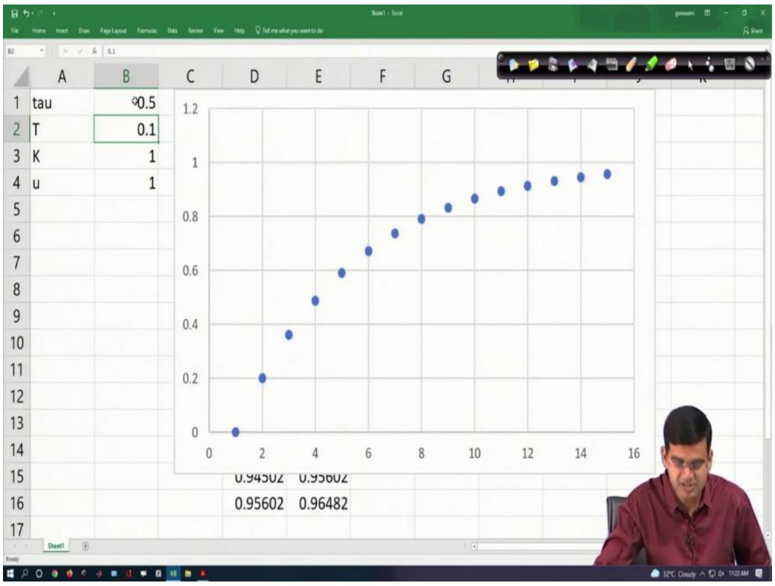
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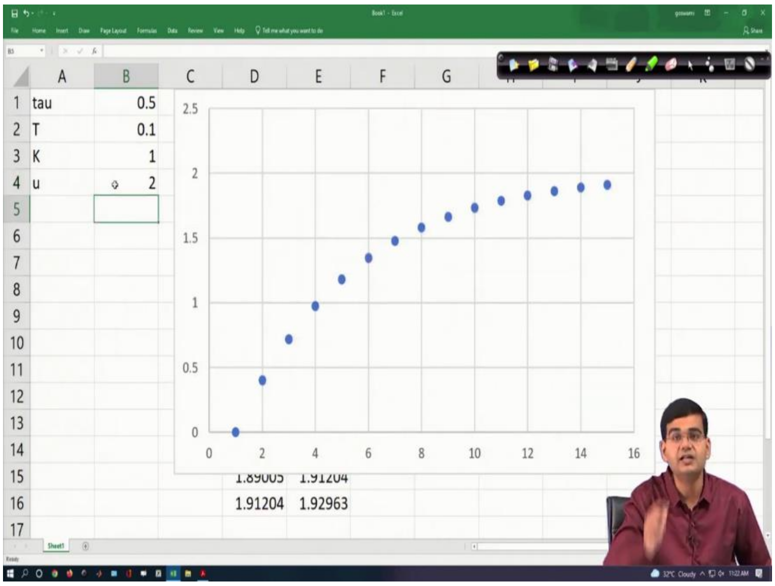
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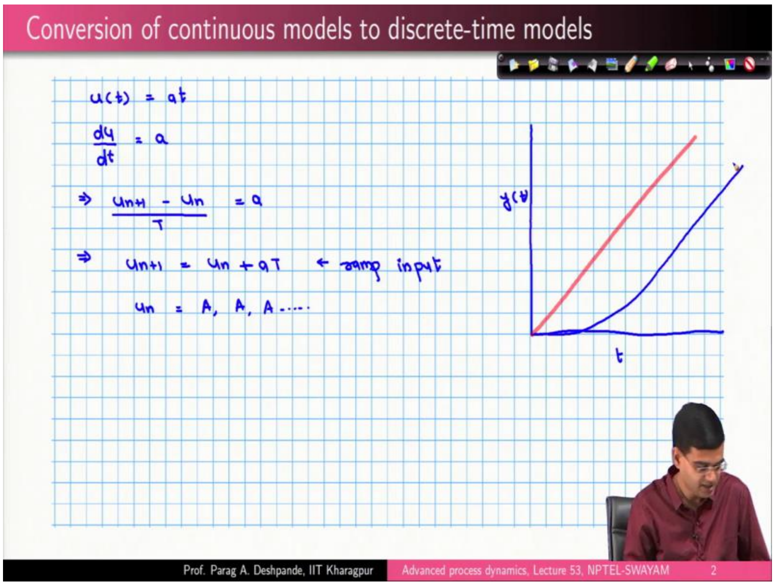
Conversion of continuous models to discrete-time models
u(t) = at
dy/dt = a
⇒ (u_{n+1} − u_n)/T = a
⇒ u_{n+1} = u_n + aT ← ramp input
u_n = A, A, A.....
y(t)
t
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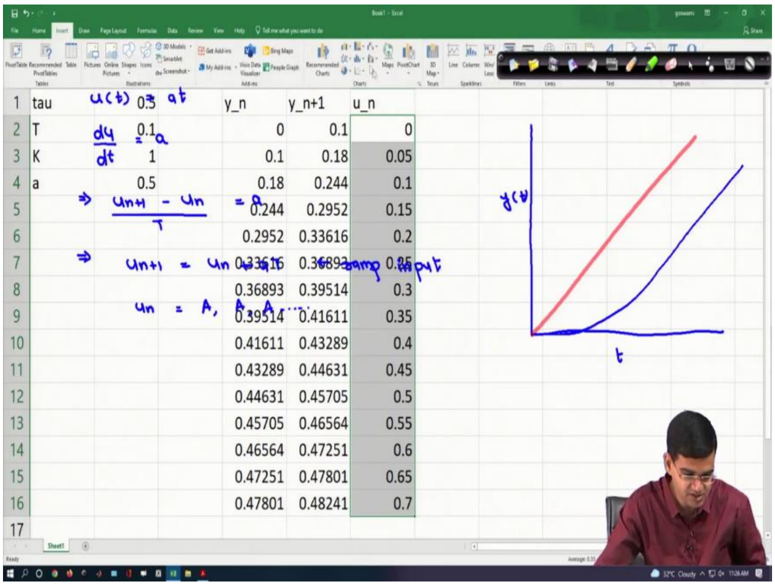
2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T 0.1
K 1
a 0.5
y_n y_n+1 u_n
0 0.1 0
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0.18 0.244 0.1
0.244 0.2952 0.15
0.2952 0.33616 0.2
0.33616 0.36893 0.25
0.36893 0.39514 0.3
0.39514 0.41611 0.35
0.41611 0.43289 0.4
0.43289 0.44631 0.45
0.44631 0.45705 0.5
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0.46564 0.47251 0.6
0.47251 0.47801 0.65
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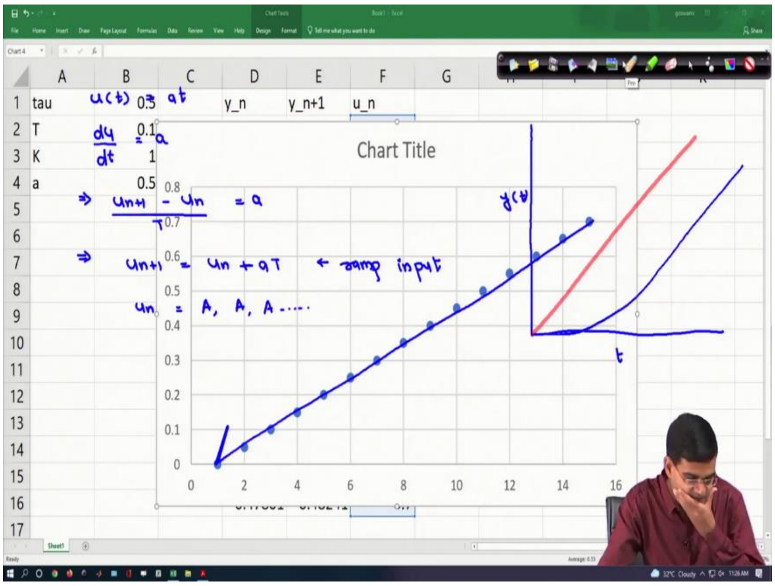
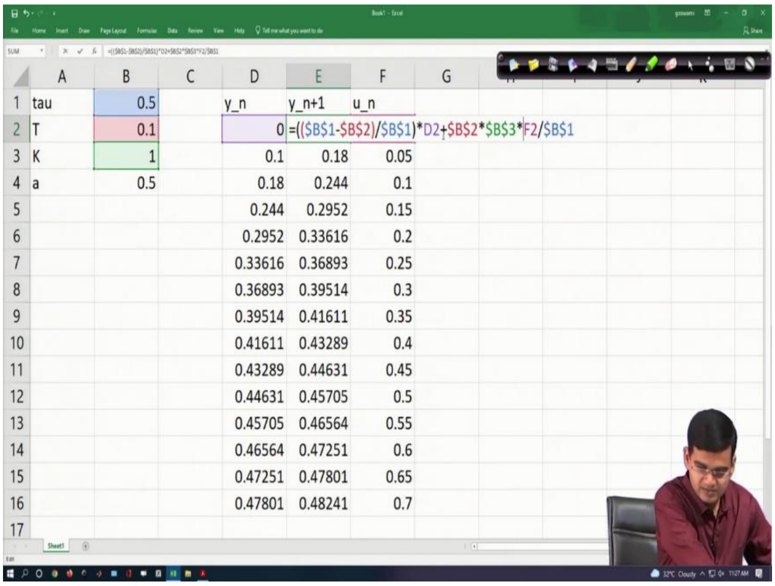
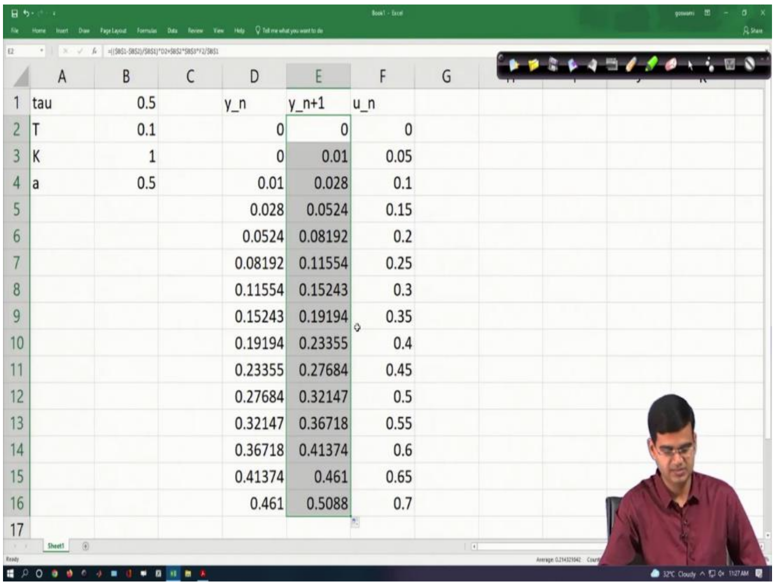
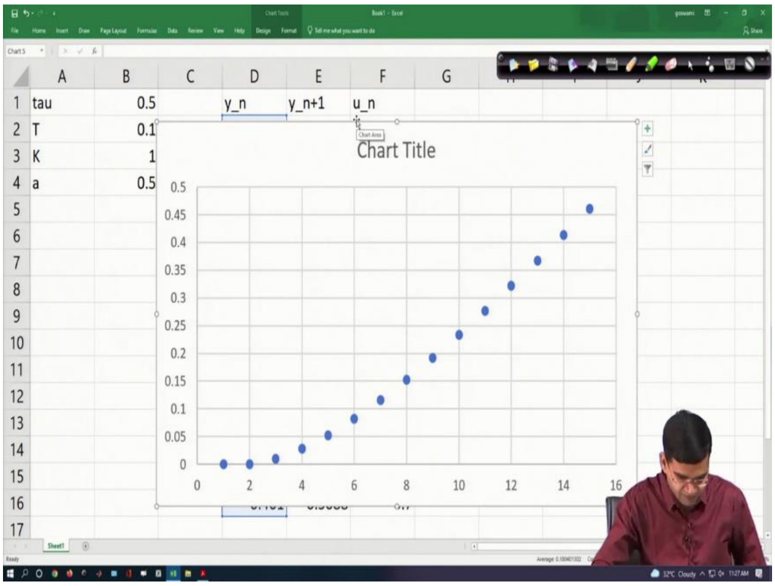


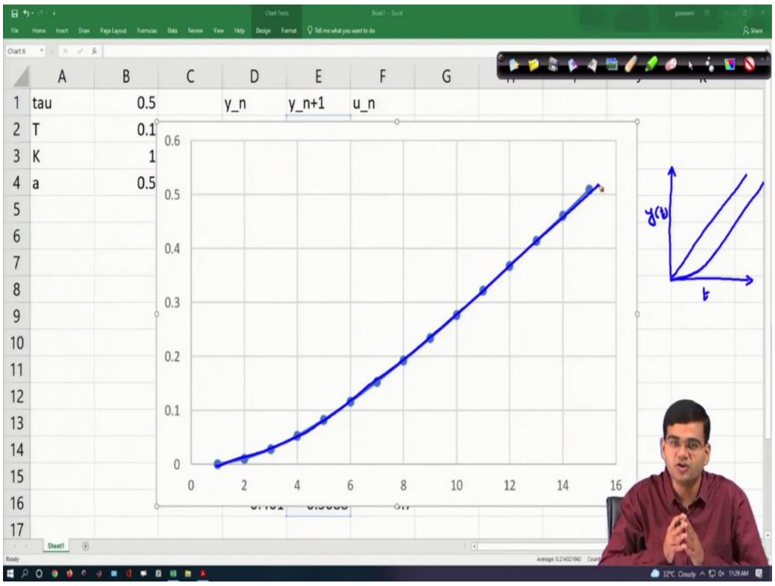


	A	B	C	D	E	F
1	tau	0.5		y_n	y_n+1	u_n
2	T	0.1		0	=(($B$1-$B$2)/$B$1)*D2+$B$2*$B$3*F2/$B$1	
3	K	1		0.1	0.18	0.05
4	a	0.5		0.18	0.244	0.1
5				0.244	0.2952	0.15
6				0.2952	0.33616	0.2
7				0.33616	0.36893	0.25
8				0.36893	0.39514	0.3
9				0.39514	0.41611	0.35
10				0.41611	0.43289	0.4
11				0.43289	0.44631	0.45
12				0.44631	0.45705	0.5
13				0.45705	0.46564	0.55
14				0.46564	0.47251	0.6
15				0.47251	0.47801	0.65
16				0.47801	0.48241	0.7
17						



	A	B	C	D	E	F
1	tau	0.5		y_n	y_n+1	u_n
2	T	0.1		0	0	0
3	K	1		0	0.01	0.05
4	a	0.5		0.01	0.028	0.1
5				0.028	0.0524	0.15
6				0.0524	0.08192	0.2
7				0.08192	0.11554	0.25
8				0.11554	0.15243	0.3
9				0.15243	0.19194	0.35
10				0.19194	0.23355	0.4
11				0.23355	0.27684	0.45
12				0.27684	0.32147	0.5
13				0.32147	0.36718	0.55
14				0.36718	0.41374	0.6
15				0.41374	0.461	0.65
16				0.461	0.5088	0.7
17						





Conversion of continuous models to discrete-time models

$$\frac{dx}{dt} = ax + bu \quad -(1)$$

$$y = cx + du \quad -(2)$$

$$\frac{x_{n+1} - x_n}{T} = a x_n + b u_n$$

$$\Rightarrow x_{n+1} = (1 + aT) x_n + bT u_n \quad -(3)$$

$$y_n = c x_n + d u_n \quad -(4)$$

$$\frac{dy}{dt} = c\frac{dx}{dt} + d\frac{du}{dt}$$

$$\Rightarrow \frac{y_{n+1} - y_n}{T} = \frac{c}{T}(x_{n+1} - x_n) + \frac{d}{T}(u_{n+1} - u_n)$$

$$\Rightarrow \frac{y_{n+1} - y_n}{T} = c(a x_n + b u_n) + \frac{d}{T}(u_{n+1} - u_n)$$

$$\frac{y_{n+1} - y_n}{T} = ac x_n + \left(bc u_n - \frac{d}{T} u_n\right) + \left(\frac{d}{T}\right) u_{n+1}$$

$$x_n = \frac{y_n - d u_n}{c}$$

$$\Rightarrow \frac{y_{n+1} - y_n}{T} = a(y_n - d u_n) + \left(bc - \frac{d}{T}\right) u_n + \left(\frac{d}{T}\right) u_{n+1} \quad -(5)$$

So, we were looking at various steps which are involved in analysis of discrete time response of systems. In the previous lecture, we looked how we can sample a continuous system to obtain a discrete time system, and how we can convert discrete time output from the controller back to a continuous signal. The only step which is left is the conversion of a continuous time model to a discrete time model.

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So, let us look into this particular aspect. Now, we will study the conversion of continuous models to discrete time models.

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And let us also understand that we are doing this with specific relevance to the response or the dynamic response of the system. So, we would, we have already looked into conversion of models from continuous time to discrete time, this particular effort would be dedicated to have a system such that we can analyze the response, the dynamical response of the system. So, let us first look into very general first order system.

So, the system is given by

$$a_1 \frac{dy}{dt} + a_0 y = bu$$

The equation is in deviation variable form, so we know that everything is normalized. So, at time t = 0, you introduce the disturbance so y starts from 0. So, to convert it to a discrete time form, let us make small changes, let us write it as

$$\frac{a_1}{a_0}\frac{dy}{dt} + y = u \dots\dots\dots (1)$$

We are doing it for convenience to club the quantities to physically important variables.

Mathematically, this same procedure can be done without this division as well. So, when the equation is cast in this particular form, we know that we can write this as

$$\tau \frac{dy}{dt} + y = Ku$$

Where,

$$\tau = \frac{a_1}{a_0}$$

τ is the time constant associated with a system and

$$K = \frac{b}{a_0}$$

K is the static gain of the system. So, now, let us convert the equation $\tau \frac{dy}{dt} + y = Ku$ to a discrete time equation.

And we will use the definition of the derivative as follows. I will write

$$\frac{dy}{dt} = \frac{y_{n+1} - y_n}{T}$$

So, why is this the definition? We are converting time in discrete domain. So, I see that my sampling of the signal takes place in this manner that I have zero and this is my time. So, the next sampling would be done at capital T. So, Δt of time is capital T, here. Further I will have 2T and so on.

So, therefore, division is by capital T, capital is T is an interval in which the sampling is done. And y_{n+1} is the intensity of the signal at the next instant given then get the intensity of the signal is y_n at the current instant. So, if this is the case, then I can write my dynamical equation as

$$\frac{\tau}{T}(y_{n+1}) - \frac{\tau}{T}\, y_n + y_n = Ku_n$$

Now, u is your forcing function that may also be a function of time. So, therefore, the intensity of your forcing function at a given instant of time, so that u_n. So, this will be

$$\frac{\tau}{T}(y_{n+1}) = \left(\frac{\tau}{T} - 1\right) y_n + K u_n$$

Or

$$y_{n+1} = \frac{T}{\tau}\left(\frac{\tau - T}{T}\right) y_n + \frac{T}{\tau} K u_n$$

which finally gives me the model equation as

$$y_{n+1} = \left(\frac{\tau - T}{\tau}\right) y_n + \left(\frac{TK}{\tau}\right) u_n \dots \dots \dots (2)$$

This is my discrete time model for a first order dynamical continuous equation. Now, if I compare equation (1) with its dynamical analog discrete time analog equation (2).

What I can see here is that I already have the dynamical discrete system in such a form that I get already an idea of the evolution of the system. So, the whole point of doing this analysis from the day 1 is to determine the evolution, the time evolution of the system. And when you have the dynamical equation, you do have an idea about the time evolution but in an indirect manner that you first need to explicitly solve the dynamical equation for the dynamical variable and then you can have an idea about dynamics.

But the dynamical equation in the discrete form itself has y_{n+1} that means, the intensity of the signal at next instant of time as a function of its intensity at right in the beginning. So, therefore, the form of equation which is there in the discrete time is quite convenient to get the evolution. Let us see, if we can have an idea about the evolution of the system with the help of this particular equation.

So, we said that if I have first order equation and u(t) the continuous time forcing function as a step function of magnitude A, a step input of magnitude A, then I know that

$$y(t) = AK(1 - e^{-\frac{t}{\tau}})$$

This was the solution which are obtained in one of the previous lectures.

So, now, what I can do is I can plot y(t) versus t. So, I have this equation y(t) as a function of t. And what I am going to do is, I know that if this is A then, for K=1, unity gain, my response is going to look something like this. Asymptotically the response curve will reach A, this is what we solved. And this is for continuous domain. Can we look into this equation, equation number (2) and see if we get a similar solution?

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So, let us try to do this. My equation here is, so the parameters associated with my system are τ, T, K and u and for a step input u would be simply a constant. So,

$$u_0 = u_1 = u_2 = \cdots u_n = u_{n+1} = A$$

So, let me assign some values, let me say that my time constant is 1, the sampling time we said that in the previous lecture that the sampling time must be smaller compared to the time constant so that you get the correct dynamical characteristics. So, let me make the sampling time as 0.1. Let us consider the unit again and unit step input, everything is 1. So, now what I have here is y_n and y_{n+1}.

I will get rid of the text so that you can see it clearly, let me just punch in the formula. So, this is equal to

$$y_{n+1} = \left(\frac{\tau - T}{\tau}\right) y_n + \left(\frac{TK}{\tau}\right) u_n$$

Let me give some so initial value of y_n ......will be 0.

Let me now get rid of the text and let us see if this is the formula which I get, if I used correct formula. This is the formula. And what I need to do is I need to make all of these quantities, constant. So, everything with B would be a constant. And then what was our recipe to propagate the system, we used to make y_1 as y_{n+1}.

So, let me see if I can propagate the system like this, yeah. Then let me simply plot this. I will insert draw a plot and let me insert a scatterplot. And what do you see here? Well, you do see that, so, let me make one more change. What you see here is that you start with 0 0 and you move here.

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Well, I can as well do one thing, I can make one more change, I can propagate the system further, and then what I can do is I can plot, I can make this plot further. So, let me draw this entire plot. And what I see here is this, that I start with 0 0 and I get a trend which looks like this. And what is on the y axis, this is y. What is on the x axis, this is t, t_n, y_n. Is this what we also got in the continuous domain? Yes, in fact, we did. The plot looked like this. Except that it used to come asymptotically to 1. Now, it also is reaching towards 1.

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Now, we can do one thing, we can change the parameters of the system. And let me make the time constant here a little smaller, so that the dynamics becomes faster. And if I do that, you see that the response has shifted up, I can make it even smaller 0.04. Sorry, I need to change the time constant. So, I will keep this same, reduce the time constant to 0.8. The responses become closer, I will make it a very small-time constant 0.5 and then you see, you are actually reaching very close to 1. So, asymptotically you will reach 1 and you can play around.

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Now, instead of unit input step input if I put 2, what is going to happen now, instead of 1 it is asymptotically reaching 2. So, all the features remain exactly the same. So, in fact, we have one way to convert this continuous time model, discrete time model and the qualitative features actually are captured correctly. Now, in this particular case, we took the example of a step function.

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Now, let us see what if we have a function which is a ramp function. So, I have a ramp input. So, I have

$$u(t) = at$$

What would be the discrete time analog of this? How would you do that? Well, let us see, if we can get it from here, I can write this as

$$\frac{du}{dt} = a$$

from where I can write

$$\frac{u_{n+1} - u_n}{T} = a$$

In other words,

$$u_{n+1} = u_n + aT$$

This is the ramp input.

What was the step input? The step input was un is equal to A, A, A and so on. It was just a constant value for every n. So, now, what we can do is, we can plug in this formula in the previous analysis. Before we do that, let us see what was our response to this kind of input. So, in continuous domain, it was y(t) and this here is t. So, our input looked like this. This is the ramp input. And our output, please refer to one of our previous lectures you will see that the output look like this, input and output.

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So, let us plug in this formula in the previous analysis. So, what we do is this let us get rid of this plot and then instead of u = 2, what you have to do is you have to specify A. And let me specify as say 0.5, the gradient which I have. So, now what will happen to the formula I need to input the value of un. So, this is u_n. And what will be u_n, initially $u_n = 0$. And what will happen to u_{n+1}?

So, $u_{n+1} = u_n + aT$, you see here u_n has changed, u_{n+1} has changed. So, what we will do is this let me make B4 and B2 constants. So, B4 and B2 would be constants. And how would un evolve with time, I can simply do this, see. So, if you want to see, if you want to ensure whether un is really a straight-line ramp function just draw this function, you see it is a straight-line function, the straight-line function. So, now, what I need to do is I need to change my formula to where I need to change un. So, let me get rid of everything here.

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And now what I will do is in this formula y_{n+1}, what I will do is I will not use this B4, instead of B4 what I will use is F2, into F2. And let me do this. Let us see what happens. Now, I have a function which looks like this. And let me plot the function. And see what you get. There is one small change which you would like to do, see the reason why you got two zeros here is because this is un it has to be un plus 1. So, everything has to be shifted by 1.

Otherwise, you can see this as the time evolution of the system, this is a better way to see. So, I see the time evolution of the system like this. So, let me plot it like this to make it look very clear. What was our continuous time domain response? We drew the continuous time domain response is this that y(t) versus t there would be an initial curve and then there would be a straight line which would be parallel to your ramp input.

You get the same analysis here. Here is an initial curve and then you go and make a perfect straight line. So, therefore, you can say that the discrete time model which you developed here actually captures the correct characteristics and we have verified it not only using step input, but we have also verified it using a ramp input. In both the cases you get the same response as what you were previously.

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So, now, if we are sure that we have some kind of a recipe to determine the discrete time analog of a continuous input, can we look into the input output model and develop a single input single output model. So, our system in that case would be

$$\frac{dx}{dt} = ax + bu \dots\dots\dots (1)$$

our dynamical equation and

$$y = cx + du \dots\dots\dots (2)$$

This is our first order input output model input be u output be y.

I would like to develop a corresponding discrete time model. So, from equation (1). I can write

$$\frac{x_{n+1} - x_n}{T} = ax_n + bu_n$$

from where I can write

$$x_{n+1} = (1 + aT)x_n + bu_n \dots\dots\dots (3)$$

So, let me refer this to as equation (3). In this particular case, I have not done the initial rearrangements. Mathematically, it is all correct, there is not an issue as far as the steps are concerned.

Similarly, for equation number (2), I can write this straight away as

$$y_n = cx_n + du_n \dots\dots\dots (4)$$

Now, this should suffice except that in equation number (4). I have x_n which is unnecessarily come up here as the dynamical variable. I may want to get rid of it. So, let us see if I can do something. So, from equation number (2) I can write

$$\frac{dy}{dt} = c\frac{dx}{dt} + \frac{du}{dt}$$

And therefore, this can be written as

$$\frac{y_{n+1} - y_n}{T} = \frac{c}{T}(x_{n+1} - x_n) + \frac{d}{T}(u_{n+1} - u_n)$$

I can write this as

$$\frac{y_{n+1} - y_n}{T} = c(ax_n + bu_n) + \frac{d}{T}(u_{n+1} - u_n)$$

Now, let us see, can I make some rearrangements, what I can do is this, I can write this as

$$\frac{y_{n+1} - y_n}{T} = acx_n + \left(bcu_n - \frac{d}{T}u_n\right) + \frac{d}{T}u_{n+1}$$

I still have x_n with me, but I can use the equation number (4). From equation number (4), I have

$$x_n = \frac{y_n - du_n}{c}$$

Now, I have y_n and I have u_n.

So, from here I can write

$$\frac{y_{n+1} - y_n}{T} = a(y_n - du_n) + \left(bc - \frac{d}{T}\right)u_n + \frac{d}{T}u_{n+1} \dots\dots\dots (5)$$

Now, see what we have here I have y_{n+1} the value of the output function at next instant of time.

And let me refer this to as equation number (5).

So, as we did this analysis previously, you can have for a ramp function, $u_{n+1} = u_n + aT$ at or you can have any type of input function. So, u_{n+1} can be determined. And then you can plug in u_n, u_{n+1}, y_n into this function giving you a single input single output model in discrete time domain., which means, you have y_{n+1} as a function of y_n, as a function of u_n and as a function of u_{n+1}.

So, this way we saw that one can develop an input output as well as any other kind of analogous discrete time model from the continuous domain modeling. What now, we would be interested in is knowing the response of the system. For this we will require a special technique called Z transforms. So, from next lecture onwards, what we will do is we will start analyzing Z transforms of discrete time variables and see how we can use Z transforms for analyzing the response of discrete time systems. Till then, goodbye.